Physics II: Electromagnetism

PH 102

Lecture 10

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Potentials: Further discussion

The primary task of electrostatics is to find the electric field of a given stationary charge distribution. In principle, this purpose is accomplished by Coulomb's law in the form of the equation

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\imath}}{\imath^2}\rho(\vec{r}\,')d\tau'$$

Unfortunately, integrals of this type may be difficult to calculate for many charge configurations except a few simple ones.

Occasionally we can get around this by exploiting symmetry and using Gauss's law, but ordinarily the best strategy is first to calculate the potential, V, given by a somewhat more tractable equation:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\imath}\rho(\vec{r}\,')d\tau'$$

Still, even this integral is often too tough to handle analytically. Moreover, in problems involving conductors ρ itself may not be known in advance; since charge is free to move around, the only thing we control directly is the total charge (or perhaps the potential) of each conductor.

So, what do you think is the way out of this situation?

Potentials: Further discussion

Recall the Poisson's equation:

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$$

which together with appropriate boundary conditions is equivalent to

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathcal{r}}\rho(\vec{r}\,')d\tau'.$$

Very often, we are interested in finding the potential in a region where $\rho = 0$.
In this case Poisson's equation reduces to Laplace's equation:

$$\nabla^2 V = 0$$

In Cartesian coordinate

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Our aim is to solve this equation in various circumstances

Note: If $\rho = 0$ everywhere then of course $V = 0$, but that's not what we meant. There may be plenty of charges everywhere, but we are confining ourselves to places where there is no charge.

A quick discussion on Differential Equation

A typical differential equation in single variable is given with an interval and boundary conditions

$$y''(x) = \frac{d^2y}{dx^2} = 0; \quad a \leq x \leq b$$

$$\text{Boundary conditions}: \quad \begin{aligned} y(a) &= \alpha \\ y(b) &= \beta \end{aligned}$$

Before applying the boundary conditions, the general solution of a differential equation is expressed in terms of arbitrary constants.

Particular solution of a differential equation is obtained from the given boundary conditions.

An ordinary differential equation (ODE) is an equation containing a function of one independent variable and its derivatives. Ex: The equation given above.

A partial differential equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives. Ex: Laplace's equation!

A quick discussion on Differential Equation

Let us start with $$y''(x) = \frac{d^2y}{dx^2} = 0; \quad a \leq x \leq b$$

Various boundary conditions lead to different results.

Boundary Condition:	Solution	Nature of Solution
No condition (General Solution)	$y = mx + c$	Solution exist; Not unique
$y(a) = \alpha$	$y = m(x - a) + \alpha$	Solution exist; Not unique
$y(a) = \alpha\,,\ y(b) = \beta$	$y = \frac{\beta - \alpha}{b - a}x + \frac{\alpha b - \beta a}{b - a}$	Solution exist; Unique
$y'(a) = \alpha, y(b) = \beta$	$y = \alpha(x - b) + \beta$	Solution exist; Unique
$y'(a) = \alpha,\ y'(b) = \beta$	No solution	Solution does not exist

The goal is to know the Existence of solution / Uniqueness of solution for a differential equation

A quick discussion on Differential Equation

Now, in general if the function y depends on more than one variable, then the differential equation becomes partial differential equation.

Wave Equation

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0$$

Schrodinger Equation

$$i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}, t) + V(\vec{r}) \psi(\vec{r}, t)$$

Poisson's Equation

$$\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right) = -\frac{\rho}{\epsilon_0}$$

The general solution to PDEs does not contain just two arbitrary constants, neither it contains any finite number despite the fact that it is a second-order equation.

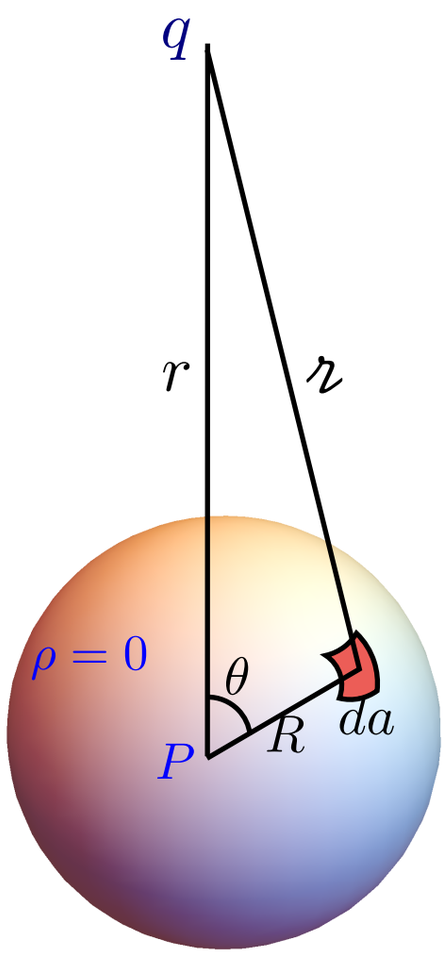
Since, now the boundaries contain infinitely many points and boundary conditions may become more complicated.

We will focus only on those types of PDEs which will be of interest to us in the discussion of electrostatics, mainly the Laplace's equation: $\nabla^2 V = 0$

Laplace's equation in three dimensions

Properties of the electrostatic potential: A Theorem:

Consider the electrostatic potential $V(r)$ in a region free of charge. (There can of course be charges outside this region producing the V. Show that $\bar{V}$, its average value over a sphere of any radius R equals $V(0)$, its value at the centre. (Note that this sphere is a mathematical surface and lies within the charge free region)



Let us calculate the average of the potential over a spherical surface of radius R due to a single point charge q located outside the sphere. We place the centre of the sphere such that q lies on the z axis. The potential on the surface of the sphere is:

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathcal{r}} \quad \text{where,} \quad \mathcal{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta}$$

Therefore, the average of V over the spherical surface

$$\bar{V} = \frac{1}{4\pi R^2}\frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\frac{q}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}}R^2\sin\theta d\theta d\phi$$

$$= \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int_0^{\pi}\frac{q}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}}\sin\theta d\theta$$

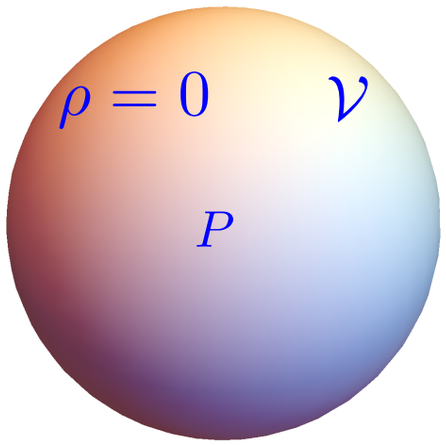
$$= \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int_{-1}^{1}\frac{q}{\sqrt{R^2 + r^2 - 2Rrx}}dx$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{2rR}[(r+R) - (r-R)] = \frac{1}{4\pi\epsilon_0}\frac{q}{r} = V(0) = V(P)$$

Properties of the electrostatic potential

Consider a charge free region $\mathcal{V}$. Let point P is inside $\mathcal{V}$. Generalisation of the previous theorem: Consider a collection of point charges $q_1, q_2, \cdots, q_n$ all placed outside the region $\mathcal{V}$. Let $V_1, V_2, \cdots, V_n$ be the potentials due to these charges respectively. Then for each charge q_i

$$V_i(0) = V_i(P) = \bar{V}_i = \frac{1}{A}\int_{\mathcal{V}} V_i(\vec{r}_i)da$$



and due to superposition principle

$$\begin{aligned} V(P) &= \sum_{i=1}^{n} V_i(P) \\ &= \sum_{i=1}^{n} \bar{V}_i \\ &= \frac{1}{A}\int_{\mathcal{V}} \left[\sum_{i=1}^{n} V_i(\vec{r}_i)\right] da = \bar{V} \end{aligned}$$

Corollary:

Let $\mathcal{V}$ be a charge free region. There can be no maximum or minimum in electrostatic potential in $\mathcal{V}$.

An important corollary

- Recall: $\vec{\nabla}.(A\vec{\nabla}B - B\vec{\nabla}A) = (A\nabla^2 B - B\nabla^2 A)$; A, B scalar fields.

- Using Gauss's divergence theorem: $\int_{\mathcal{V}} (A\nabla^2 B - B\nabla^2 A) d\tau = \oint_{\mathcal{S}} (A\vec{\nabla}B - B\vec{\nabla}A).d\vec{a}$

- Choose $A = \frac{1}{|\vec{r}|}$ and $B = V$, i.e. B represents a solution of Laplace's equation and A satisfies:

$$\nabla^2 A = \nabla^2 \frac{1}{|\vec{r}|} = -4\pi\delta^3(\vec{r})$$

- If we assume the surface $\mathcal{S}$ to be a sphere of radius R,

$$4\pi \int_{\mathcal{V}} V(\vec{r})\delta^3(\vec{r}) d\tau = \oint_{\mathcal{S}} \left(\frac{1}{R}(\vec{\nabla}V) + \frac{\hat{r}}{R^2} V \right).d\vec{a}$$

$$V(\vec{r} = 0) = \frac{1}{4\pi R^2} \oint_{\mathcal{S}} V(R, \theta, \phi) da$$

$\frac{1}{R}(\vec{\nabla}V) \to 0$

Since

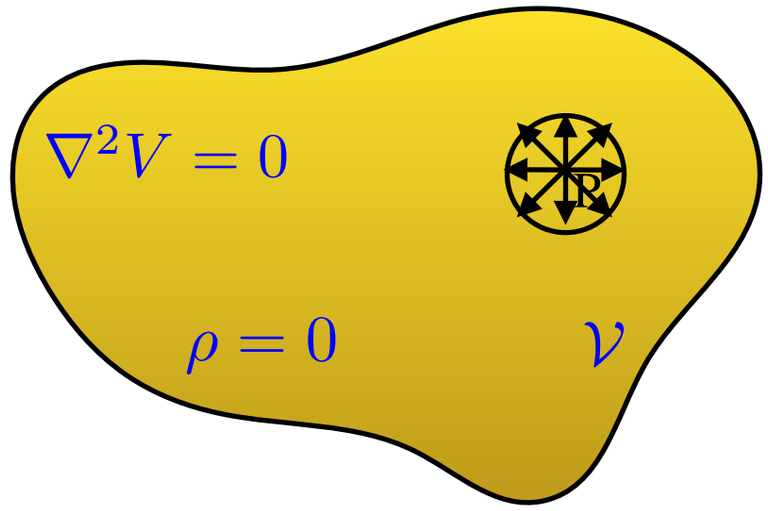
$$\nabla^2 V = 0 \longrightarrow \vec{\nabla}.(\vec{\nabla}V) = 0 \xrightarrow[\text{theorem}]{\text{Using divergence}} \int_{\mathcal{S}} (\vec{\nabla}V).d\vec{a} = \int_{\mathcal{V}} \nabla^2 V d\tau = 0$$

If V satisfies Laplace's equation inside a sphere of radius R, then the value of V at the origin of the sphere is the average of the value of V over the surface of the sphere.

Properties of the electrostatic potential

Let $\mathcal{V}$ be a charge free region. There can be no maximum or minimum in electrostatic potential in $\mathcal{V}$.

Proof: Since there is no charge inside $\mathcal{V}$: $\nabla^2 V = 0$



But note that the charges can be present outside as well as on the surface of $\mathcal{V}$.

We know that potentials will be there inside, on the surface, as well as outside $\mathcal{V}$.

Let us choose a point P inside and say that the potential at P has a maxima.

If potential V has a maxima at P, then V must be decreasing everywhere surrounding the point P.

Now, we know that in whichever directions potential decreases, the electric field is directed in those directions. $\vec{E} = -\vec{\nabla} V$.

So, if potential is decreasing in the surroundings of P, then electric field lines will always come out of a Gaussian surface chosen around P!

That means there is a positive flux coming out of P, but that requires the presence of positive charge at P. But $\rho = 0$ inside $\mathcal{V}$. There is no charge.

Our assumption was wrong. There can be no maximum or minimum in electrostatic potential in $\mathcal{V}$.

Boundary conditions and Uniqueness theorem:

Laplace's equation does not by itself determine V. Suitable boundary conditions must be supplied.

Q. What are appropriate boundary conditions?

We have seen that this is easy in one dimension. The general solution is of the form $V=mx+c$. This contains two arbitrary constants m and c and we therefore require two boundary conditions.

Two and three dimensional cases are not easy, we confronted PDEs. The boundaries are not two points (like in one dimension) any more. They are surfaces and a surface contains infinitely many points and hence the boundary conditions become more complicated.

But, by now your intuition may tell you that V is uniquely determined by its value at the boundary.

The proof that a proposed set of boundary conditions will suffice is usually presented in the form of **Uniqueness Theorem**. There are many such theorems in electrostatics. We will discuss a few.

First Uniqueness Theorem

The solution to Laplace's Equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface $\mathcal{S}$ and $\rho = 0$.

Proof: Suppose the solution is not unique in the volume.

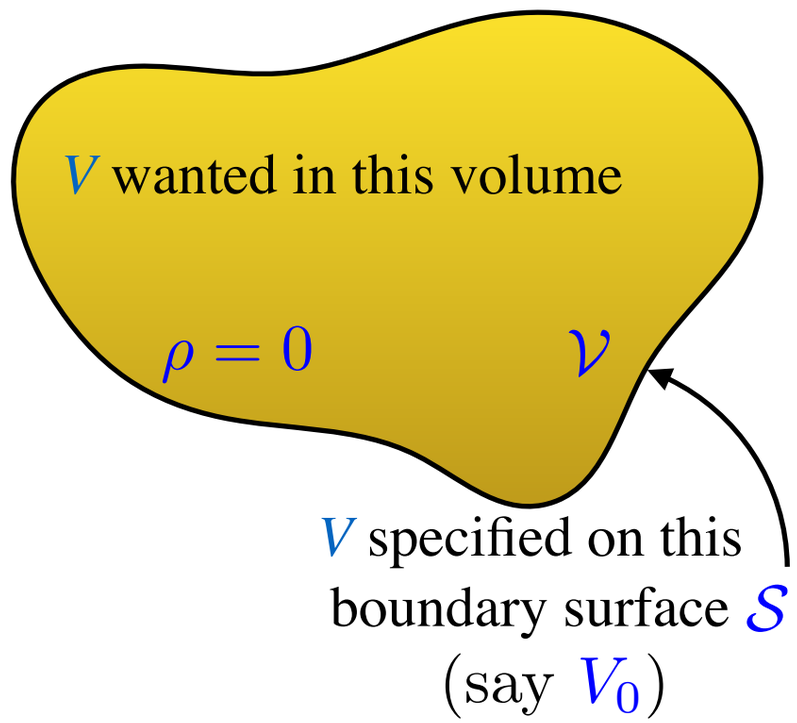
Say, there are two solutions to the Laplace's equation: $V_1,\ V_2$

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0 \quad (\text{in } \mathcal{V})$$

Both of which assumes specific values on the boundary surface.

i.e. V_1, V_2 takes value V_0 on the surface.

We want to prove that they must be equal inside the volume too.



The trick is to look at their difference: $V_3 \equiv V_1 - V_2$

Since V_3 is a linear superposition of V_1 and V_2, it must obey Laplace's equation:

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0 \quad (\text{in } \mathcal{V})$$

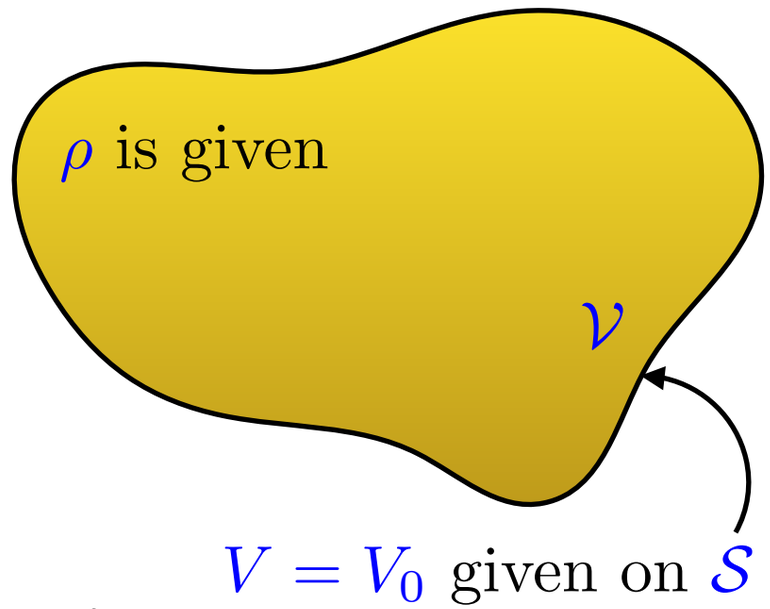
What is the status of V_3 on the surface?

$V_3 = 0$ on the surface because $V_1 = V_2 = V_0$ on the surface and $V_3 \equiv V_1 - V_2$

So, we have $V_3 = 0$ on $\mathcal{S}$ and also $\nabla^2 V_3 = 0$. But, Laplace's equation allows no local maxima or minima. So maxima and minima of V_3 are both zero. Therefore V_3 must be zero everywhere inside $\mathcal{V}$. $\implies V_1 = V_2$.

First Uniqueness Theorem: Improvement

The potential in a volume $\mathcal{V}$ is uniquely determined if (a) the charge density ρ throughout the region, and (b) the value of V on the boundaries, are specified.



Proof: Suppose the solution is not unique in the volume $\mathcal{V}$.

Say, there are two solutions to the Laplace's equation: $V_1,\ V_2$

Now, we have $\nabla^2 V_1 = -\frac{\rho}{\epsilon_0}$ and $\nabla^2 V_2 = -\frac{\rho}{\epsilon_0}$

$V_1 = V_0$ and $V_2 = V_0$ on the surface $\mathcal{S}$.

Once again the difference $V_3 = V_1 - V_2$ satisfies Laplace's equation

$$\text{so}\quad \nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0} = 0 \quad \text{throughout } \mathcal{V} \text{ and } V_3 = 0 \text{ on } \mathcal{S}$$

$$\text{Recall: } \vec{\nabla}.(V_3 \vec{\nabla} V_3) = (\vec{\nabla} V_3)^2 + V_3 \vec{\nabla}^2 V_3$$

$$\text{Gauss's theorem} \implies \int_{\mathcal{V}} \{(\vec{\nabla} V_3)^2 + V_3 \vec{\nabla}^2 V_3\} d\tau = \oint_{\mathcal{S}} V_3 \vec{\nabla} V_3 . d\vec{a}$$

$\vec{\nabla}^2 V_3$: 0 throughout $\mathcal{V}$

V_3: 0 on $\mathcal{S}$

Hence $\int_{\mathcal{V}} (\vec{\nabla} V_3)^2 d\tau = 0 \implies \vec{\nabla} V_3 = 0$ on $\mathcal{V} \implies V_3 = V_1 - V_2 = constant.$

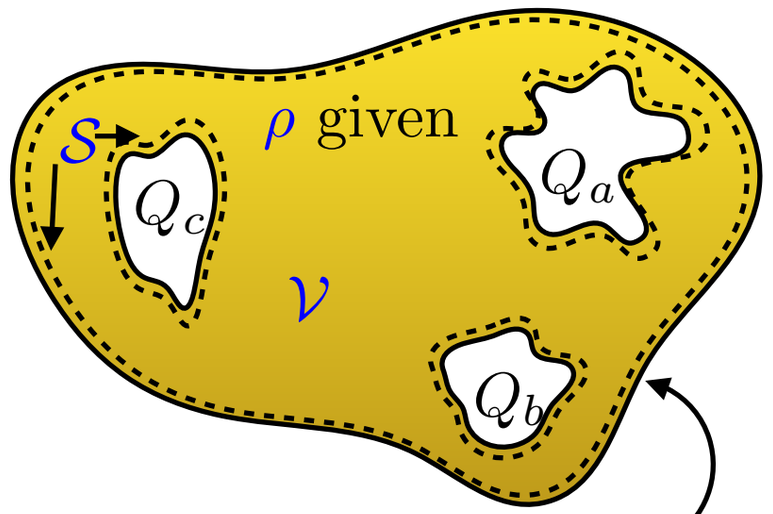
However, adding a constant to the potential does not change any physical quantities like electric field. i.e. the initial assumption was incorrect

Conductors and Second Uniqueness Theorem

In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given (The region as a whole can be bounded by another conductor, or else unbounded).

Proof:

Suppose there are two fields satisfying the conditions of the problem.



In $\mathcal{V}$: region between conductors

$$\vec{\nabla}.\vec{E}_1 = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla}.\vec{E}_2 = \frac{\rho}{\epsilon_0}$$

Both obeys Gauss's law in integral form for a Gaussian surface enclosing each conductor

$$\oint_{\substack{\text{i-th conducting} \\ \text{surface}(S_i)}} \vec{E}_1.d\vec{a}_i = \frac{Q_i}{\epsilon_0}, \qquad \oint_{\substack{\text{i-th conducting} \\ \text{surface}\,(S_i)}} \vec{E}_2.d\vec{a}_i = \frac{Q_i}{\epsilon_0}$$

Similarly, for the outer boundary $\oint_{\substack{\text{outer} \\ \text{boundary}\,(S)}} \vec{E}_1.d\vec{a} = \frac{Q_{\text{tot}}}{\epsilon_0}, \qquad \oint_{\substack{\text{outer} \\ \text{boundary}\,(S)}} \vec{E}_2.d\vec{a} = \frac{Q_{\text{tot}}}{\epsilon_0}$

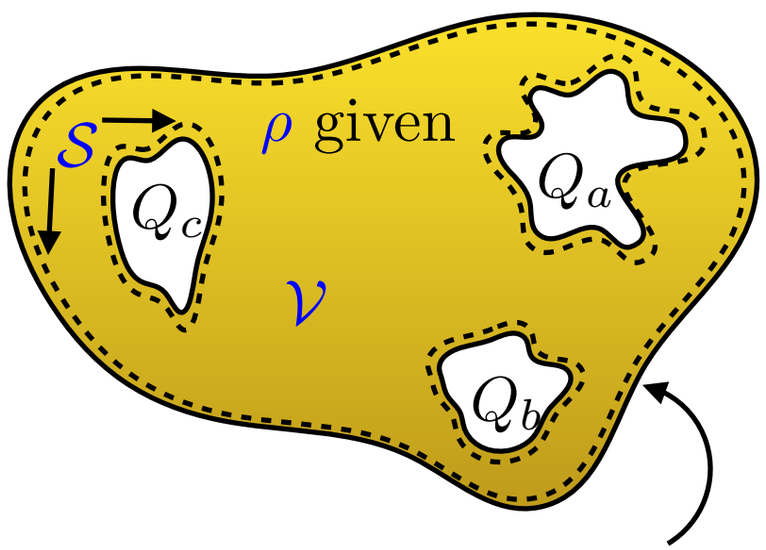
where, $Q_{\text{tot}} = \sum_{i=1}^{N} Q_i + \int \rho d\tau$

Conductors and Second Uniqueness Theorem

As before, we examine the difference $\vec{E}_3 = \vec{E}_1 - \vec{E}_2$; which obeys $\vec{\nabla}.\vec{E}_3 = 0$ in the region between the conductors and

$$\oint \vec{E}_3.d\vec{a}_i = 0 \quad \text{over each boundary surface.}$$

$$\oint \vec{E}_3.d\vec{a} = 0 \quad \text{over the whole boundary}$$



Conductors are equipotential $\implies$ if $\vec{E}_3 = -\vec{\nabla}V_3$ then V_3 is a constant over each conducting surface

If S is at infinity, then $V_3 = V_2 = V_1 = 0$ on S.

If S is a conductor, then V_3 on S is constant.

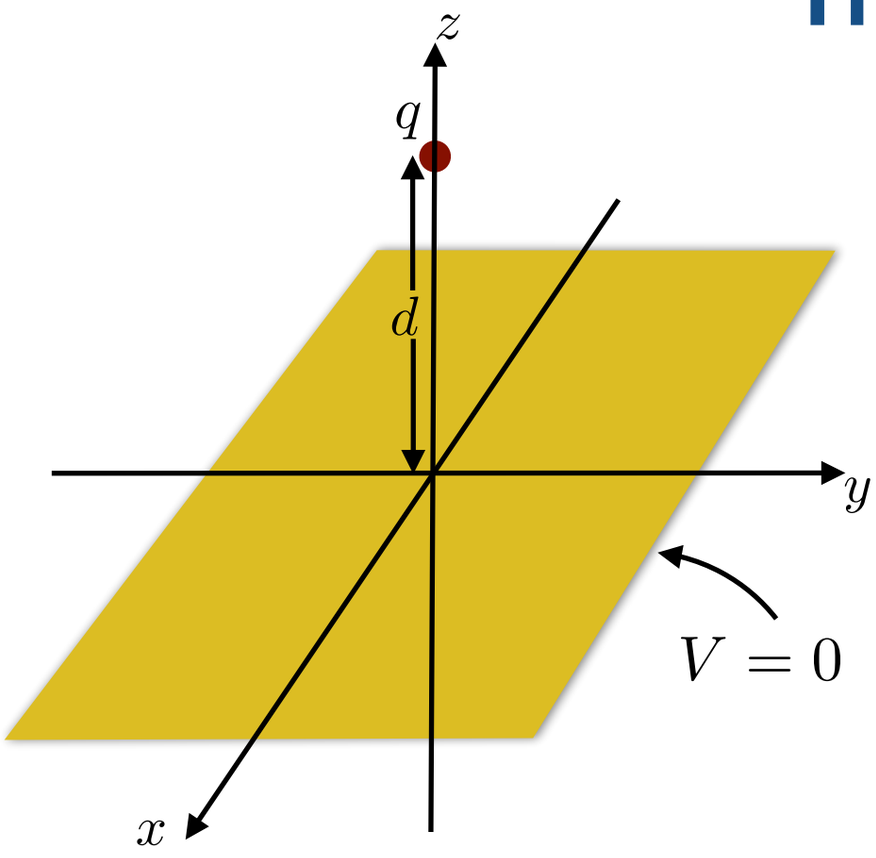
Use $\vec{\nabla}.(V_3\vec{E}_3) = V_3(\vec{\nabla}.\vec{E}_3) + \vec{E}_3.(\vec{\nabla}V_3) = -(E_3^2)$ since $\vec{\nabla}.\vec{E}_3 = 0, \quad \vec{E}_3 = -\vec{\nabla}V_3$

Using Divergence theorem: $\int_{\mathcal{V}} \vec{\nabla}.(V_3\vec{E}_3)d\tau = \oint_{\mathcal{S}} V_3\vec{E}_3.d\vec{a} = -\int_{\mathcal{V}} (E_3)^2 d\tau$

Covers all boundaries of the region

V_3 is a constant over each surface and comes out of integral. But $\int \vec{E}_3.d\vec{a} = 0 \implies \int_{\mathcal{V}}(E_3)^2 d\tau = 0 \implies \vec{E}_3 = 0 \implies \vec{E}_1 = \vec{E}_2$.

The method of images



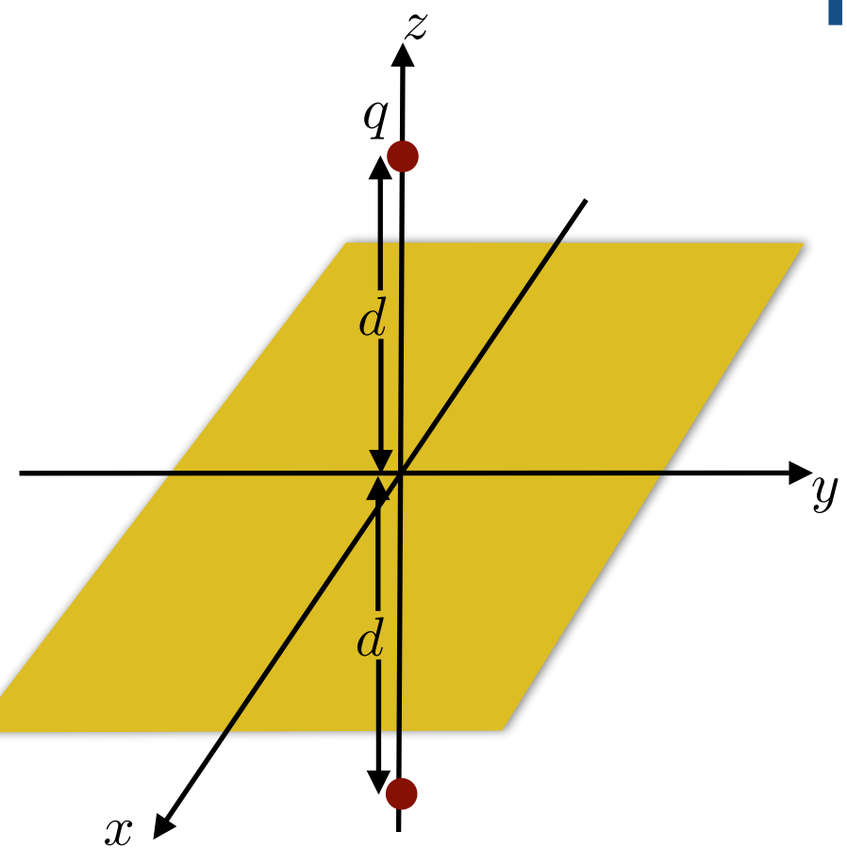
- A point charge q
- Held at a distance d above an infinite grounded conducting plane
- Q. What is the potential in the region above the plane?
- It is not $\frac{1}{4\pi\epsilon_0}\frac{q}{r}$.
- Because q will induce negative charges on the conductor.
- Total potential is due to q and induced charge on plane.

- Need to solve Poisson's equation in the region $z > 0$, with a single point charge q at $(0, 0, d)$, subject to the boundary condition:

  1. $V = 0$ at $z = 0$ (since conducting plane is grounded),
  2. $V \to 0$ far from the charge (i.e. for $x^2 + y^2 + z^2 >> d$).

- First uniqueness theorem tells us that there is only one function that meets these requirements. If by trick or clever guess we can find the function, it is going to be the answer.

The method of images



- Trick: Forget the actual problem!!
- Trick: Think of a new configuration: 2 point charges: $+q$ at $(0,0,d)$ and $-q$ at $(0,0,-d)$.
- Potential for such a configuration

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

- It follows that (1) $V = 0$ at $z = 0$; (2) $V \to 0$ for $x^2 + y^2 + z^2 >> d^2$.
- And the only charge in $z > 0$ is $+q$ at $(0,0,d)$.
- These are precisely the conditions of the original problem.
- Second configuration happens to produce the exactly same situation and boundary conditions as the first one for $z \geq 0$.
- Hence the potential for a point charge above an infinite grounded conducting plane is given by above formula.