Physics II: Electromagnetism

PH 102

Lecture 11

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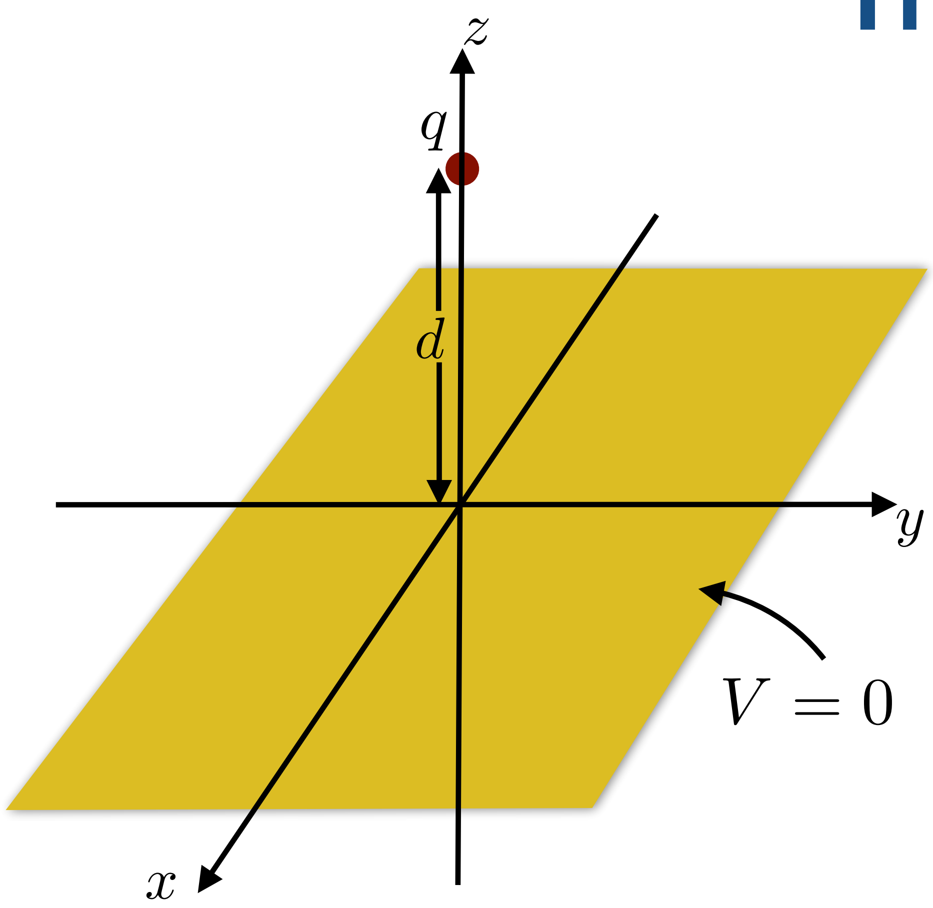
The method of images



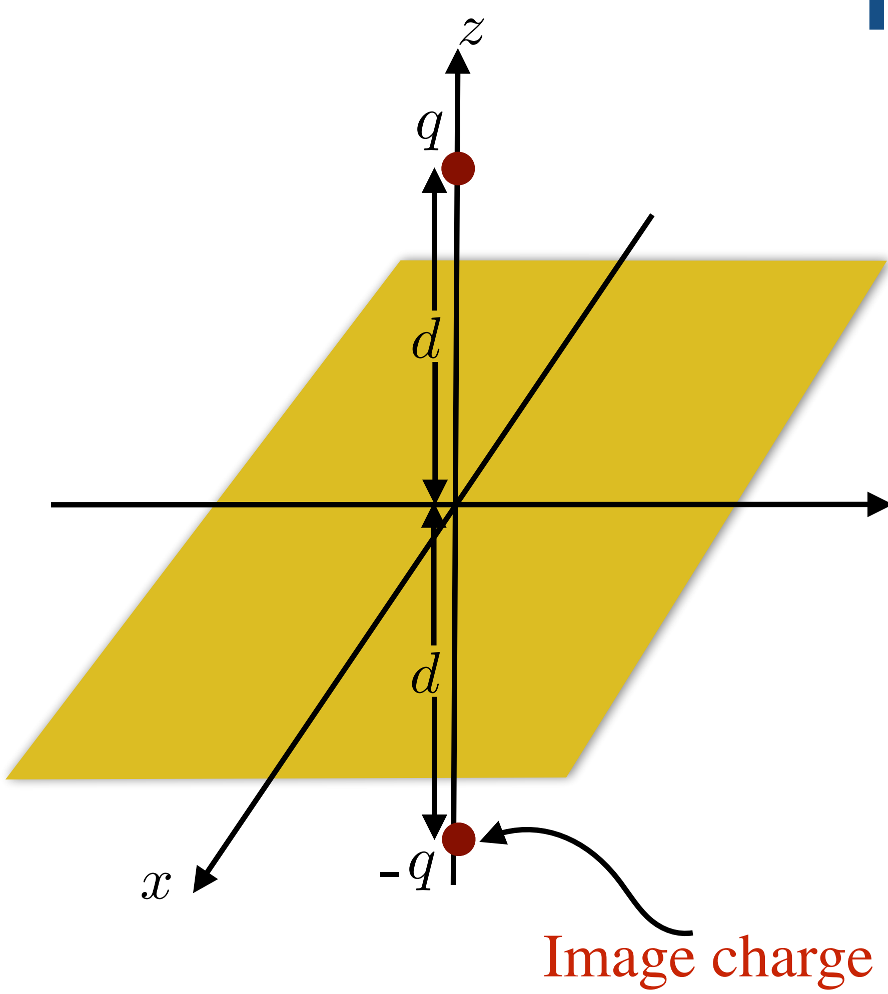
- A point charge q
- Held at a distance d above an infinite grounded conducting plane
- Q. What is the potential in the region above the plane?
- It is not $\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$.
- Because q will induce negative charges on the conductor.
- Total potential is due to q and induced charge on plane.

- Need to solve Poisson's equation in the region $z > 0$, with a single point charge q at $(0, 0, d)$, subject to the boundary condition:

  1. $V = 0$ at $z = 0$ (since conducting plane is grounded),
  2. $V \to 0$ far from the charge (i.e. for $x^2 + y^2 + z^2 >> d$).

- First uniqueness theorem tells us that there is only one function that meets these requirements. If by trick or clever guess we can find the function, it is going to be the answer.

The method of images



- Trick: Forget the actual problem!!

- Trick: Think of a new configuration: 2 point charges: $+q$ at $(0,0,d)$ and $-q$ at $(0,0,-d)$.

- Potential for such a configuration

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

- It follows that (1) $V=0$ at $z=0$; (2) $V \to 0$ for $x^2+y^2+z^2 >> d^2$.

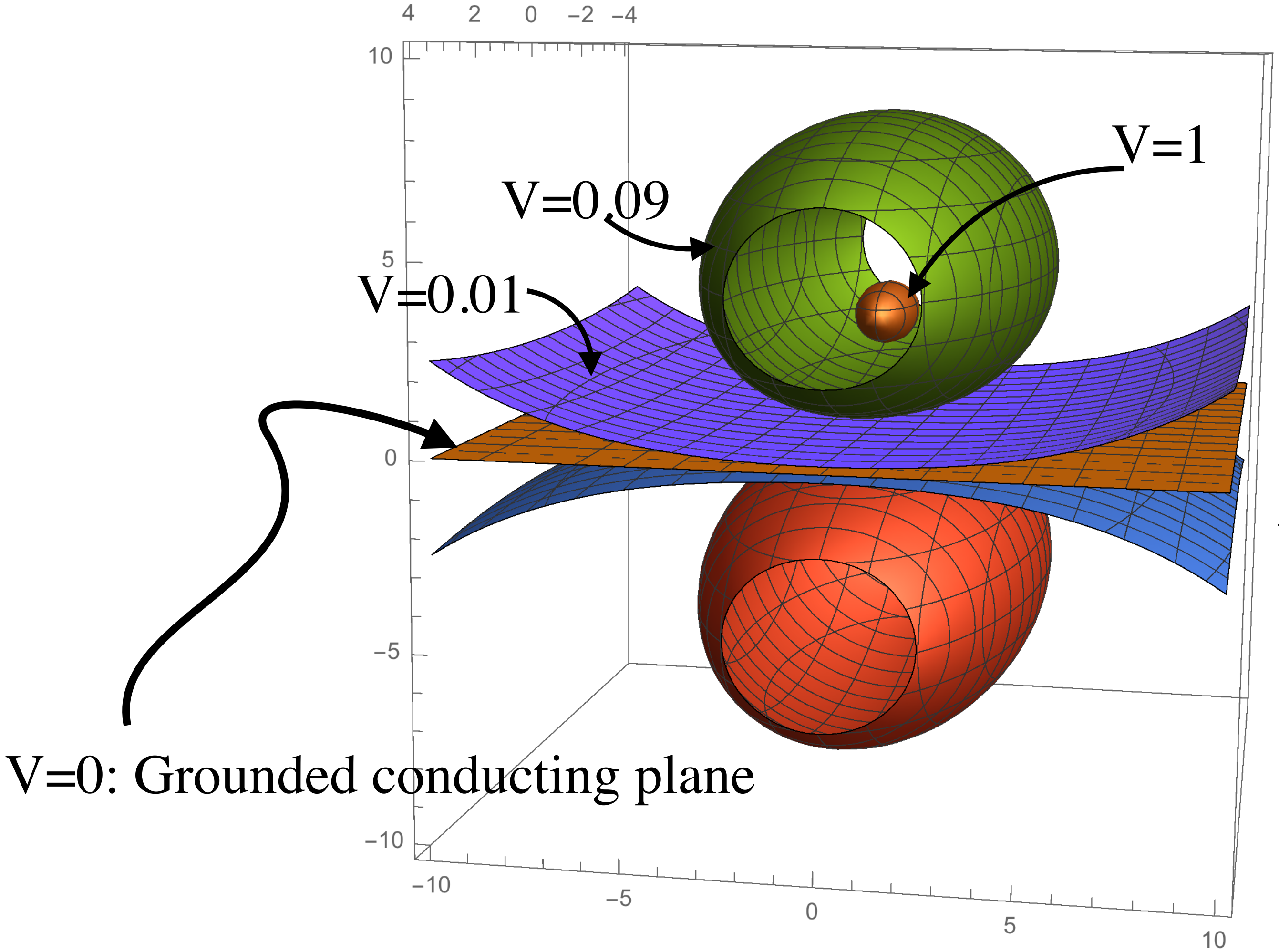
- And the only charge in $z>0$ is $+q$ at $(0,0,d)$.

- These are precisely the conditions of the original problem.

- Second configuration happens to produce the exactly same situation and boundary conditions as the first one for $z \geq 0$.

- Hence the potential for a point charge above an infinite grounded conducting plane is given by above formula.

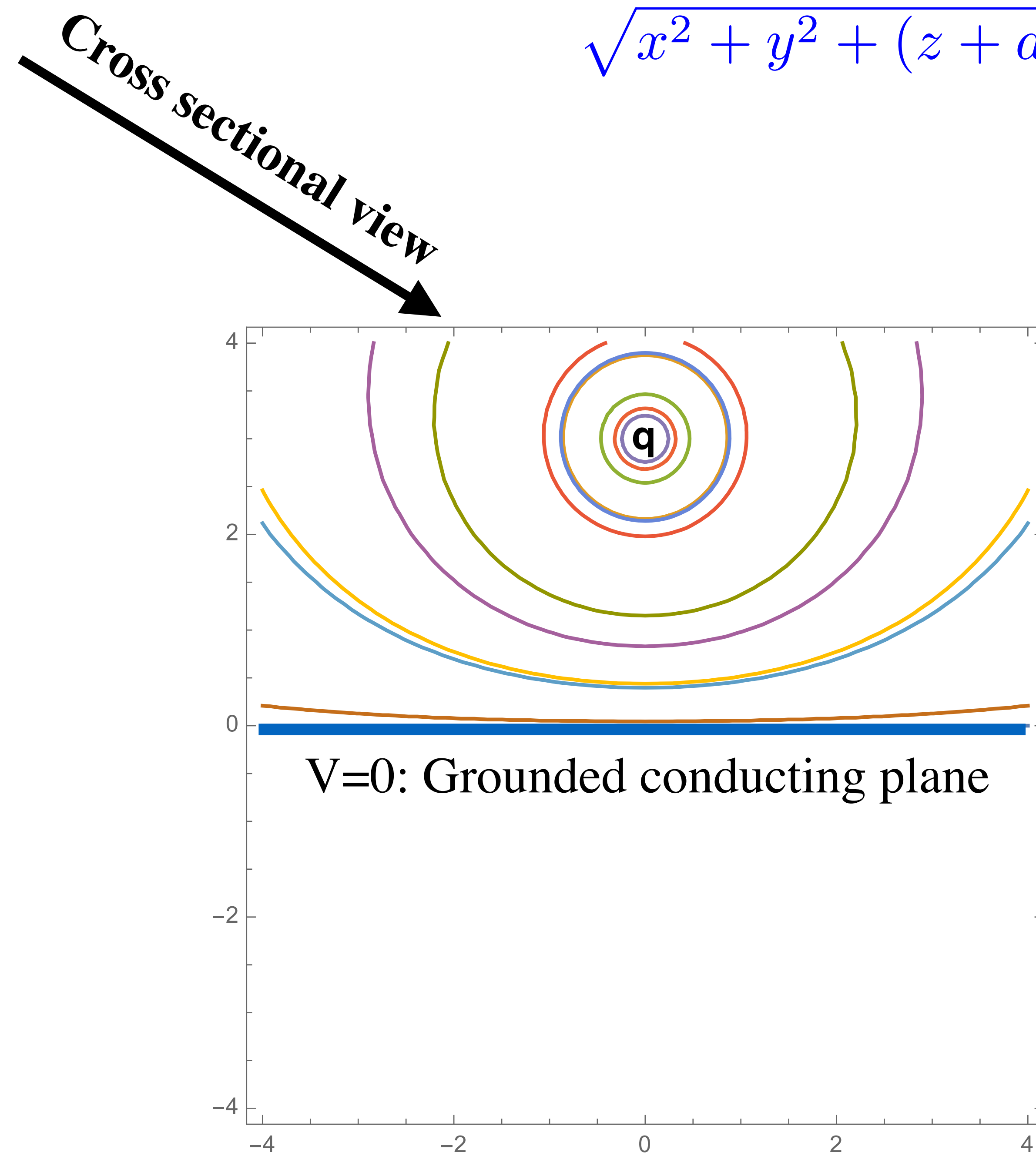
The method of images



Equipotential surfaces

z<0 is not of interest to us

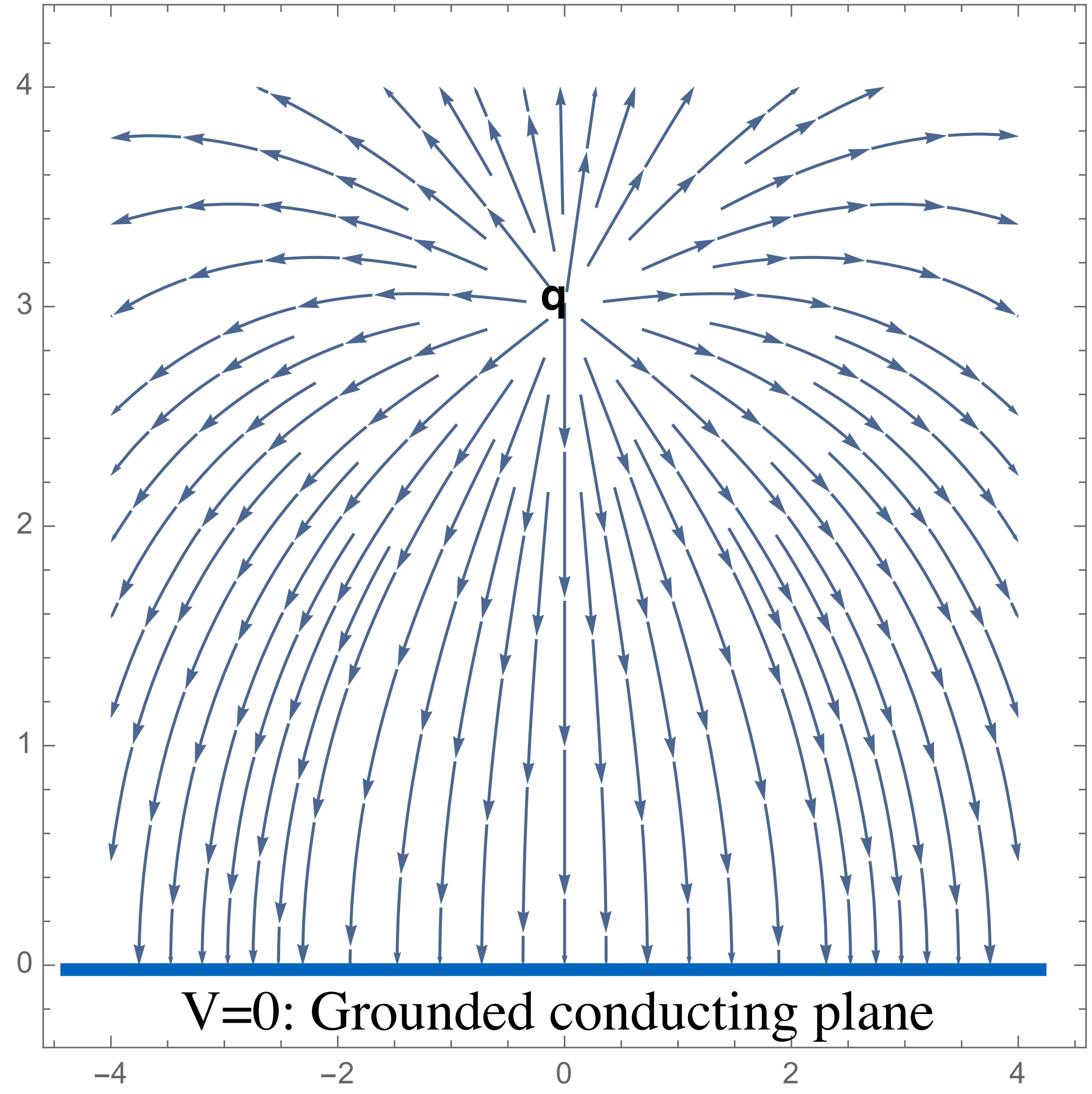
$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$



The method of images

The electrostatic field for the image charge problem:

$$\vec{E}(x,y,z) = -\vec{\nabla}V = \frac{q}{4\pi\epsilon_0}\left[\frac{x\hat{x}+y\hat{y}+(z-d)\hat{z}}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{x\hat{x}+y\hat{y}+(z+d)\hat{z}}{(x^2+y^2+(z+d)^2)^{3/2}}\right]$$



The method of images

Induced surface charge What is the surface charge induced on the conductor?

Now that we know the potential in the $z > 0$ region, we can easily work out the distribution of charges induced on the conducting plate.

Recall that the electric field immediately above a conducting surface:

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n} \implies \sigma = -\epsilon_0 \frac{\partial V}{\partial n} \implies \sigma = -\epsilon_0 \frac{\partial V}{\partial z}$$

where $\frac{\partial V}{\partial n}$ is the normal derivative at the surface, and for us, the direction of the normal is in z direction:

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0}\left[\frac{-q(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} + \frac{q(z+d)}{(x^2+y^2+(z+d)^2)^{3/2}}\right]$$

Therefore the charge density:

$$\sigma(x,y) = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0} = -\frac{\epsilon_0}{4\pi\epsilon_0}\frac{2qd}{(x^2+y^2+d^2)^{3/2}} = -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}$$

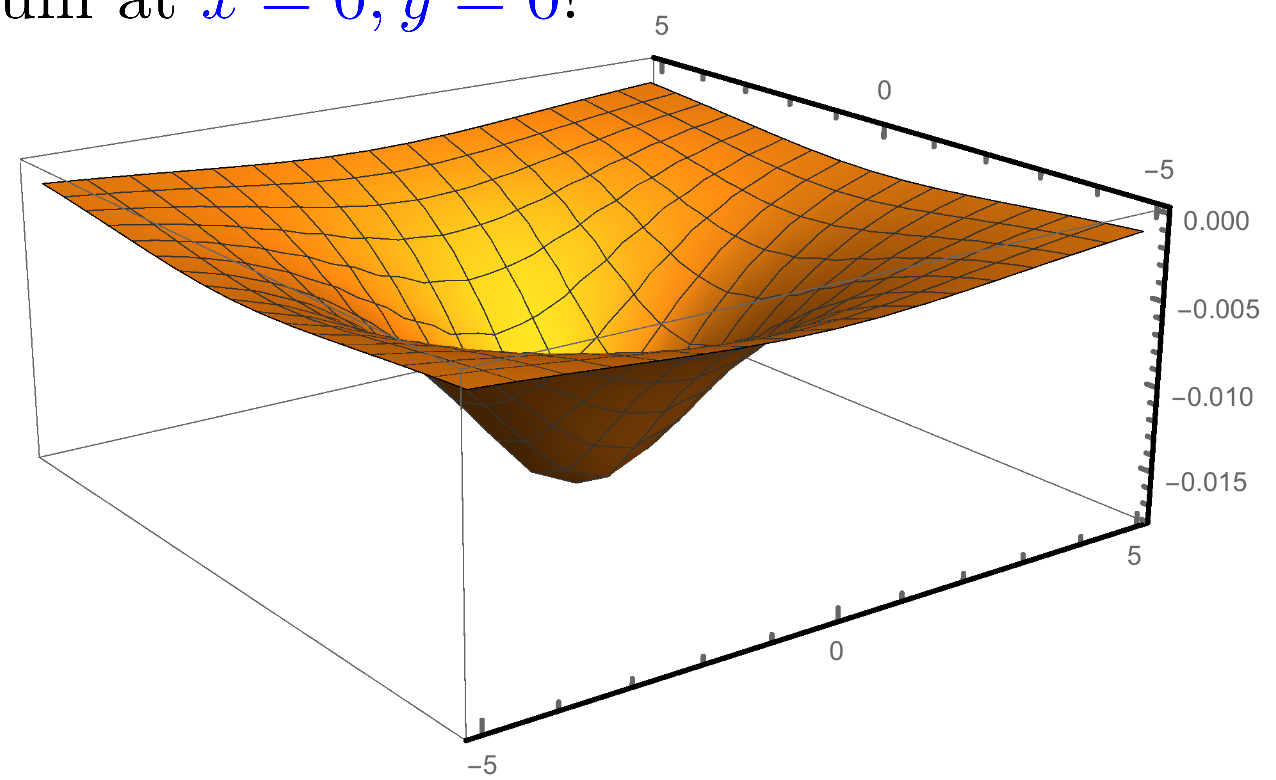
As expected, the induced charge on the conducting plate is negative.

The method of images

Induced surface charge density:

$$\sigma(x,y) = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0} = -\frac{\epsilon_0}{4\pi\epsilon_0}\frac{2qd}{(x^2+y^2+d^2)^{3/2}} = -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}$$

Induced charge is maximum at $x=0, y=0$!



So, the total charge: $Q = \int \sigma da$

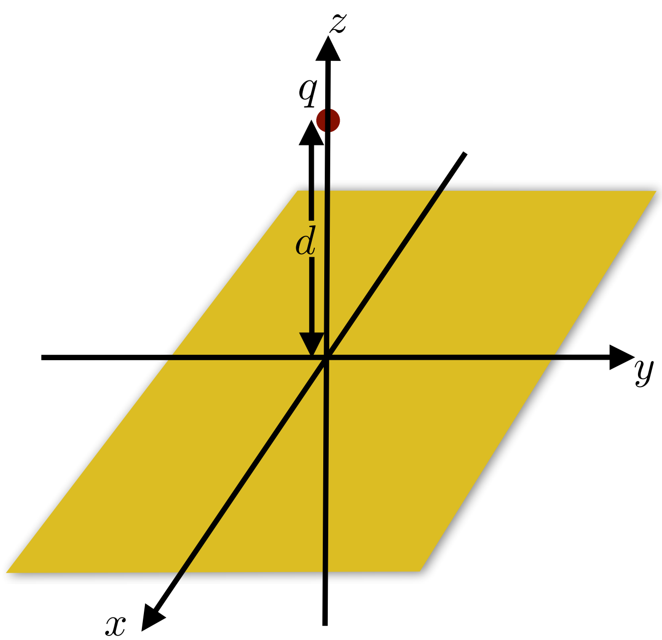
It is easier to perform the integral in polar coordinate: $r^2 = x^2 + y^2$ and $da = r dr d\phi$

$$Q = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}} r dr d\phi$$

$$= -\frac{qd}{\sqrt{r^2+d^2}}\bigg|_0^{\infty} = -q$$ EXPECTED!

This means that all the flux leaving the point charge q is actually ending on the conducting plane.

The method of images



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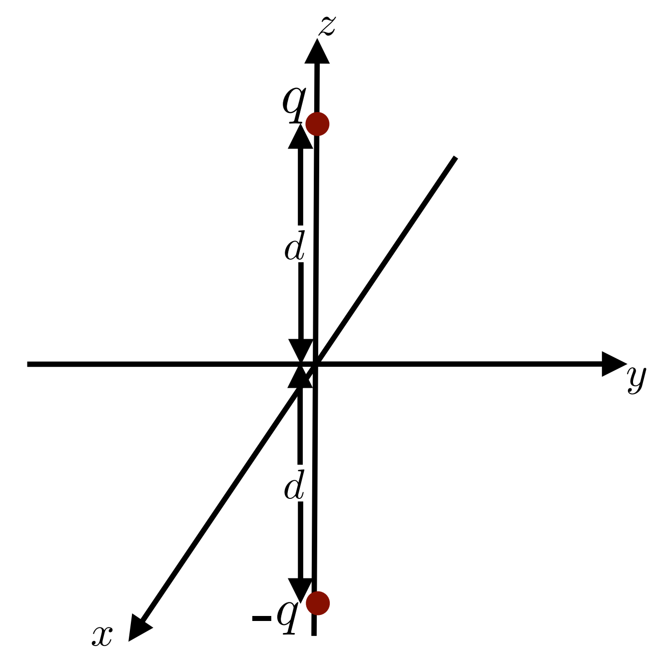


Beware: Everything is not the same in the two problems! Energy is not the same!

For point charge and the plane

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$$

For two point charges

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}$$

Energy is half than that when there are two point charges

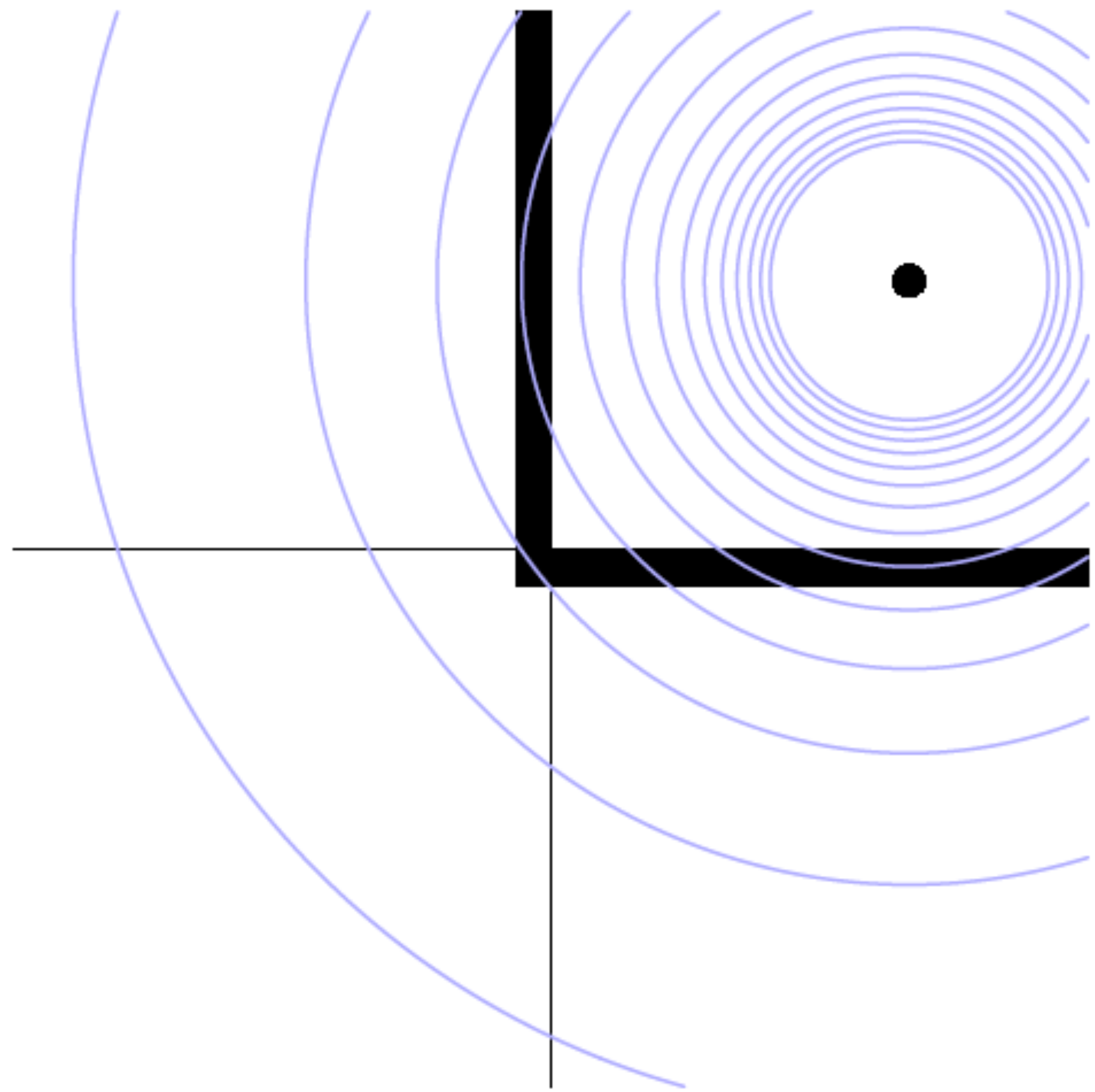
Only the upper (z>0) region contributed to the energy.

Recall

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau$$

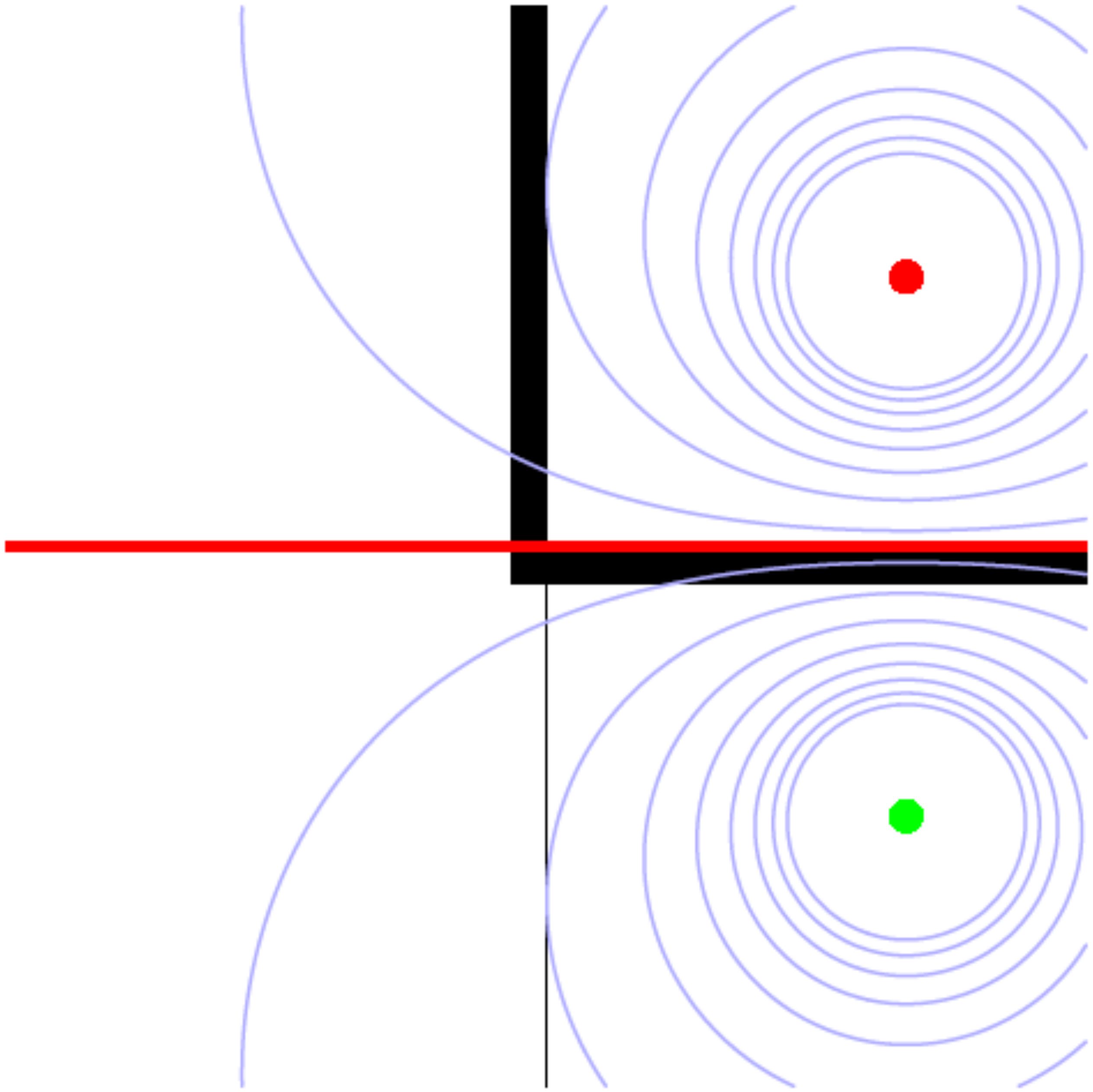
We have considered both upper (z>0) and lower (z<0) regions and by symmetry both contributed equally.

Two Infinite Grounded Conducting Planes



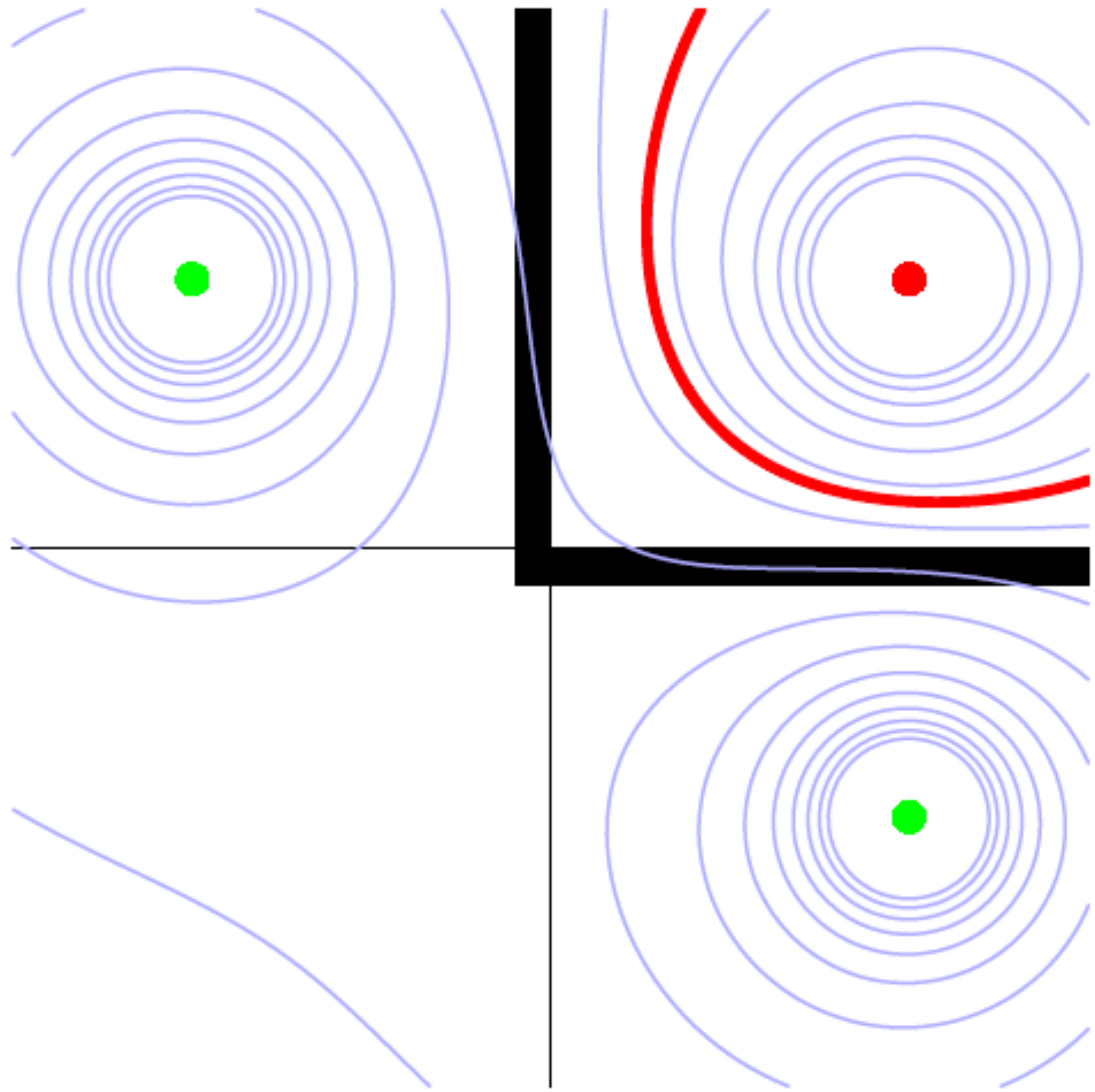
The two grounded conducting surfaces at right angles, say, *xy* & *xz*-planes and a real charge placed in the first quadrant.

Two Infinite Grounded Conducting Planes



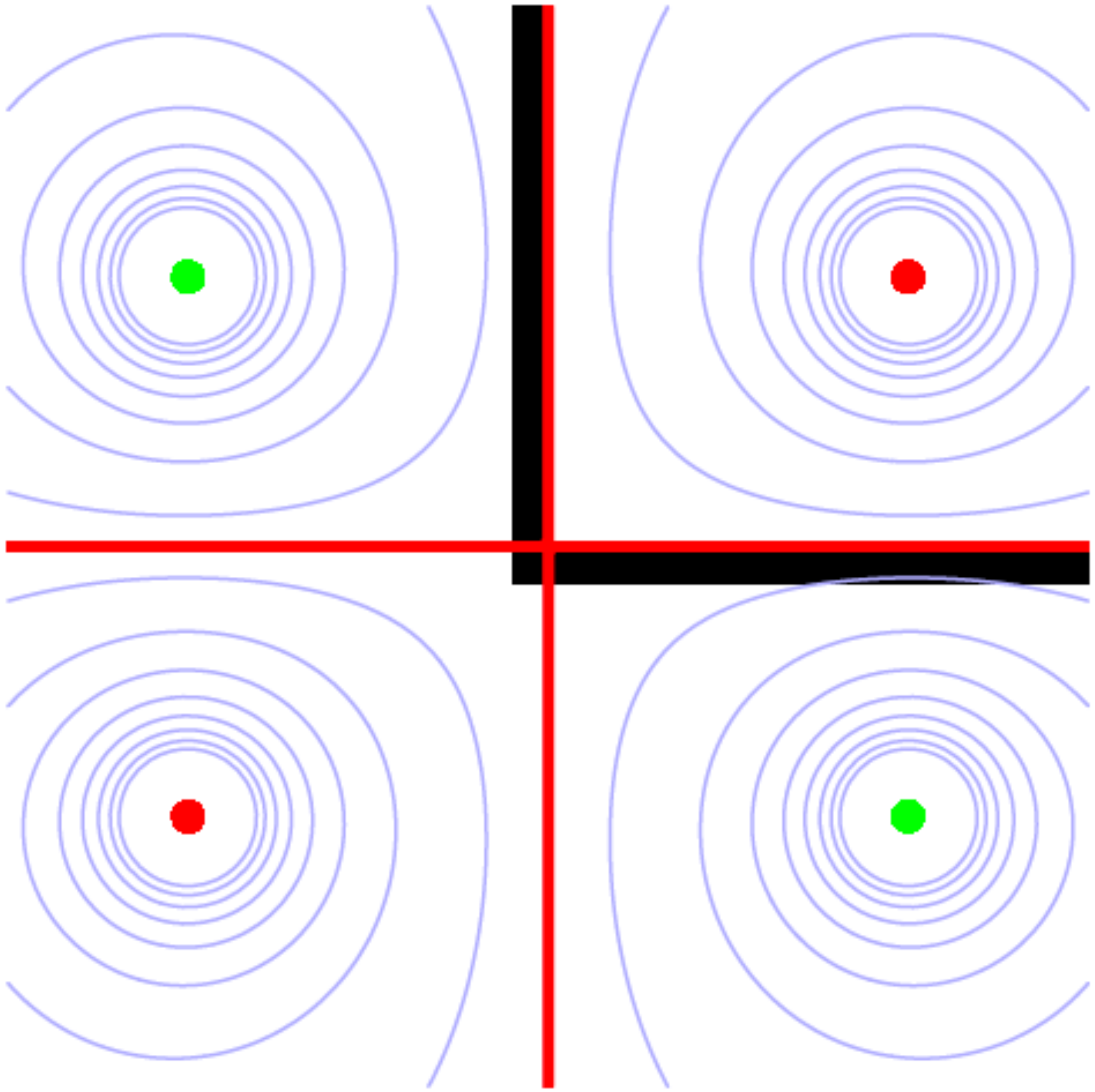
One image charge is not good enough. Red line is the zero equipotential.

Two Infinite Grounded Conducting Planes



Even two image charges are not good enough. Red line is the zero equipotential.

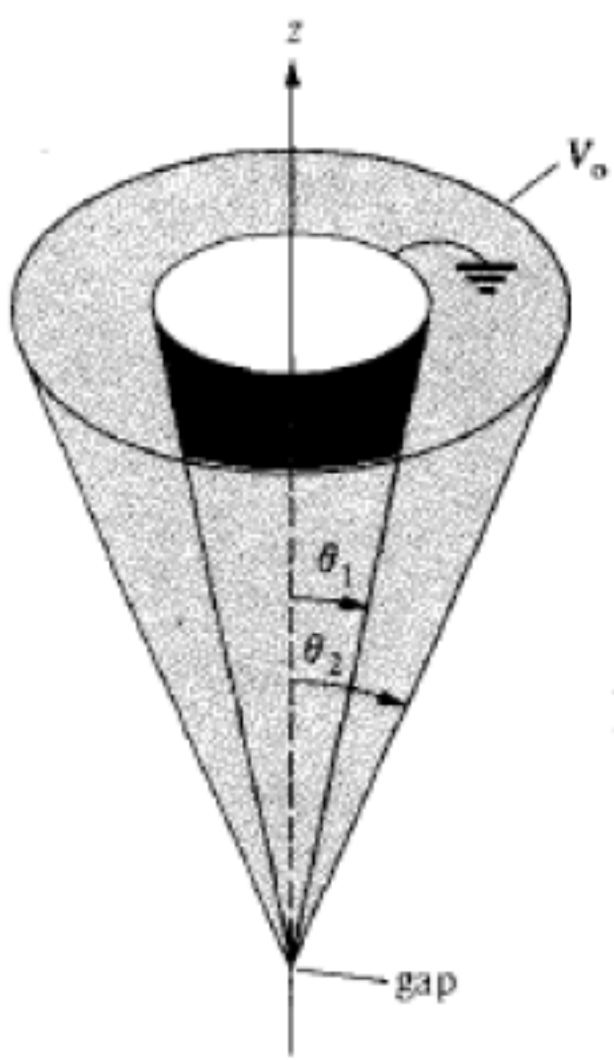
Two Infinite Grounded Conducting Planes



Must put three image charges. Red line is the zero equipotential.

Method of Direct Integration
Example

Consider two co-axial conducting cones of infinite extent with angles θ_1 and θ_2, respectively, and separated by an infinitesimal insulating gap at $r = 0$. Find V in the region between the two cones, given that the inner cone is grounded while the outer cone is kept at constant potential V_0 .



V depends only on θ, so Laplace's equation in spherical coordinates

$$\nabla^2 V = \frac{1}{r^2 \sin\theta} \frac{d}{d\theta}\left[\sin\theta \frac{dV}{d\theta}\right] = 0$$

Since $r = 0$ and $\theta = 0, \pi$ are excluded, we can multiply by $r^2 \sin\theta$

$$\frac{d}{d\theta}\left[\sin\theta \frac{dV}{d\theta}\right] = 0$$

Integrating once gives

$$\frac{dV}{d\theta} = \frac{A}{\sin\theta}$$

Integrating this results to get

$$V = A \int \frac{d\theta}{\sin\theta} = A \ln(\tan\theta/2) + B$$

We now apply the boundary conditions to determine the integration constants A and B.

$$\text{(a)}\quad V(\theta = \theta_1) = 0 \quad \rightarrow \quad 0 = A \ln(\tan \theta_1/2) + B$$

$$\boxed{B = -A \ln(\tan \theta_1/2)}$$

Hence

$$V = A \ln\left[\frac{\tan \theta/2}{\tan \theta_1/2}\right]$$

Also

$$\text{(b)}\quad V(\theta = \theta_2) = V_o \quad \rightarrow \quad V_o = A \ln\left[\frac{\tan \theta_2/2}{\tan \theta_1/2}\right]$$

$$\boxed{A = \frac{V_o}{\ln\left[\dfrac{\tan \theta_2/2}{\tan \theta_1/2}\right]}}$$

Thus

$$\boxed{V(\theta) = \frac{V_o \ln\left[\dfrac{\tan \theta/2}{\tan \theta_1/2}\right]}{\ln\left[\dfrac{\tan \theta_2/2}{\tan \theta_1/2}\right]}}$$

Solving Laplace's equation directly: Separation of Variables

Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$. The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates and maintained at a specific potential $V_0(y)$. Find the potential inside this infinite "slot". (Griffiths, Example 3.3)

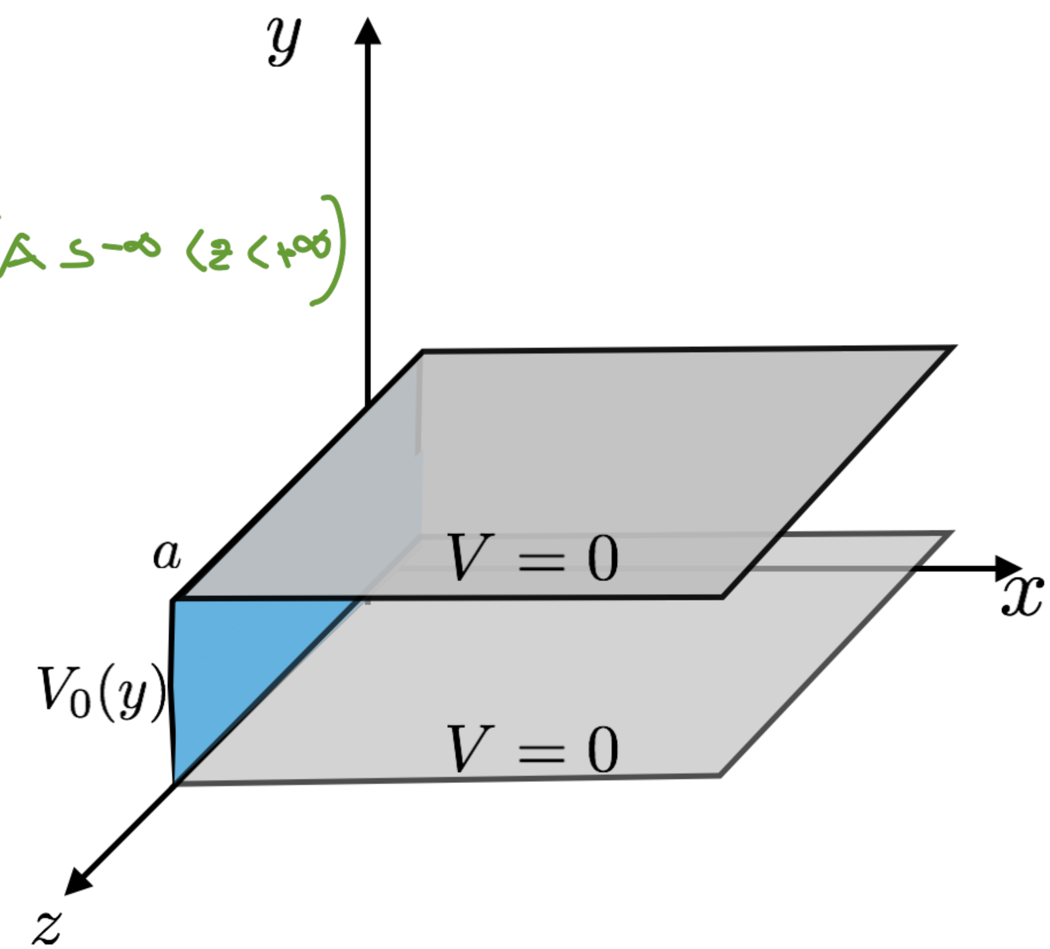
Few observations:

- There is a translational symmetry along z direction, (As $-\infty < z < +\infty$) therefore the potential must be independent of z.

- Our region of interest is $x > 0, 0 < y < a$.

- The boundary has six surfaces

$$\begin{aligned} x &= 0 \ \text{and} \ \ x = \infty \\ y &= 0 \ \text{and} \ \ y = a \\ z &= \pm\infty \end{aligned}$$



- We need to solve Laplace's eqn:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Boundary conditions: (on the surfaces)

$$\begin{aligned} V(x, y, z) &= V_0(y) \ \ \text{for } x = 0 \\ V(x, y, z) &= 0 \ \ \text{for } y = 0, \ y = a \\ V(x, y, z) &= 0 \ \ \text{for } x \to \infty \end{aligned}$$

Separation of Variables **Strategy:**

• Look for solutions of the form $V(x,y) = X(x)Y(y)$. There is no guarantee that such solutions exists. But in some cases, such separation may be possible.

• But that is absurd!

Not all solutions can be written in that product form.

Gives only a tiny subset of all possible solutions

• These kind of product solutions may not fit the boundary conditions. Apply as many conditions as possible to narrow down number of possible solutions.

• Laplace's equation is linear: $\nabla^2 V_1 = 0, \quad \nabla^2 V_2 = 0 \implies \nabla^2(\alpha V_1 + \beta V_2) = 0$

• Hence $\alpha V_1 + \beta V_2$ is also a solution to Laplace's equation. It is therefore possible that some linear combination of such solutions may fit the remaining boundary conditions.

Separation of Variables

Strategy: separate, divide and conquer!

Substitute the solution $V(x,y) = X(x)Y(y)$ into Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2}{\partial x^2}(X(x)Y(y)) + \frac{\partial^2}{\partial y^2}(X(x)Y(y)) = 0$$

$$Y(y)\frac{\partial^2 X(x)}{\partial x^2} + X(x)\frac{\partial^2 Y(y)}{\partial y^2} = 0$$

Note: Total derivative
Why?

Dividing both sides by $X(x)Y(y)$

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2}$$

Now, the LHS is a function of x only and RHS is function of y only.

This necessarily means that each side must be equal to a constant!

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} = k^2 \quad \text{(say)}.$$

We get two separate equations:

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = k^2$$

$$-\frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} = k^2$$

"Divide and rule" policy worked!

A PDE has been converted to 2 separate ODEs and ODEs are much easier to solve!

Separation of Variables

Solutions:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = k^2 \Longrightarrow X(x) = Ae^{kx} + Be^{-kx}$$

$$-\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = k^2 \Longrightarrow Y(y) = C\sin ky + D\cos ky$$

Boundary Conditions:

$$V(x,y,z) = V_0(y) \quad \text{for } x = 0$$

$$V(x,y,z) = 0 \quad \text{for } y = 0,\ y = a$$

$$V(x,y,z) = 0 \quad \text{for } x \to \infty$$

Hence the potential $V(x,y) = X(x)Y(y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$

Now, let us fit the boundary conditions: (to fix the constants A, B, C etc.)

- Since as $x \to \infty$, $V \to 0$: $\Longrightarrow$ $X(x) \to 0$, hence coefficient of e^{kx} must vanish. This means $A = 0$. Absorbing B into C and D, we get

$$V(x,y) = e^{-kx}(C\sin ky + D\cos ky)$$

- Since, at $y = 0$, $V = 0$: $\Longrightarrow$ $Y(y) = 0$: only possible if coefficient of $\cos ky$ is zero. Hence $D = 0$. $\therefore V(x,y) = Ce^{-kx}\sin ky$.

- Again, at $y = a$, $V = 0$: $\Longrightarrow$ $Y(y) = 0$, i.e. $Y(a) = C\sin ka = 0$

$$\Longrightarrow k = k_n = \frac{n\pi}{a}, \quad (n = 1, 2, 3, \cdots)$$

This gives us countably infinite number of solutions: $V_n(x,y) = C\ e^{-k_n x}\sin k_n y$

Separation of Variables

There is one more Boundary Condition, which is still unused. $V(x, y, z) \;=\; V_0(y)$ for $x = 0$

So, the situation now is the following:

• We have countably infinite number of solutions: $V_n = C\ e^{-n\pi x/a}\sin(n\pi y/a)$.

• Unless $V_0(y)$ just happens to have the form $\sin(n\pi y/a)$ for some n, we can not fit the final boundary condition at $x = 0$.

• Separation of variable has given us an infinite family of solutions (one for each n), but, none of them by itself satisfies the final boundary condition!

n	k_n	$V_n(x, y)$	$V_n(\infty, y)$	$V_n(x, 0)$	$V_n(x, a)$	$V_n(0, y)$
1	$\frac{\pi}{a}$	$e^{-\pi x/a}\sin(\frac{\pi y}{a})$	0	0	0	$\sin(\frac{\pi y}{a})$
2	$\frac{2\pi}{a}$	$e^{-2\pi x/a}\sin(\frac{2\pi y}{a})$	0	0	0	$\sin(\frac{2\pi y}{a})$
3	$\frac{3\pi}{a}$	$e^{-3\pi x/a}\sin(\frac{3\pi y}{a})$	0	0	0	$\sin(\frac{3\pi y}{a})$
..	..	..	..	..	..	..

How to incorporate the final condition $V(x, y, z) \;=\; V_0(y)$ for $x = 0$**?**

Separation of Variables

Construct a linear combination out of the solutions V_n of the Laplace's Equation:

$$V(x,y) = \sum_{n=1}^{\infty} \alpha_n V_n(x,y) = \alpha_1 V_1(x,y) + \alpha_2 V_2(x,y) + \cdots$$

Each V_n satisfies Laplace's equation separately. Therefore:

$$\nabla^2 V = \alpha_1 \nabla^2 V_1 + \alpha_2 \nabla^2 V_2 + \cdots = 0\alpha_1 + 0\alpha_2 + \cdots = 0$$

(Laplace's equation is linear!)

Exploiting this fact, we can patch together the separable solutions to construct a more general solution:

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right).$$

Still satisfies the three boundary conditions

The question: can we fit the final boundary condition by choosing the C_n's?

i.e. we must find $C_1, C_2, \cdots$ etc. such that the final boundary condition:

$$V(0,y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y)$$

is satisfied.

Can we do that?

Separation of Variables **Fourier's Trick**

Our problem is essentially solved if we can uniquely find $C_1, C_2, \cdots$ such that

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$$

Fourier Sine Series

The good news is that the existence of such a unique set of numbers C_n is guaranteed by Dirichlet's theorem: virtually any function $V_0(y)$ (even having some finite number of discontinuities) can be expanded in the above Fourier Sine Series.

Let m be a positive integer, then multiply by $\sin(m\pi y/a)$ both sides and integrate

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) = \sin\left(\frac{m\pi y}{a}\right) V_0(y)$$

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{m\pi y}{a}\right) \sin\left(\frac{n\pi y}{a}\right) dy = \int_0^a \sin\left(\frac{m\pi y}{a}\right) V_0(y) dy$$

The integral in LHS is 0 if $m \neq n$ and is $\frac{a}{2}$ if $m = n$.

Essentially all the terms in the series drop out, except the one with $n = m$!

$$\therefore\ C_m = \frac{2}{a} \int_0^a \sin\left(\frac{m\pi y}{a}\right) V_0(y) dy$$

The solution : At last!

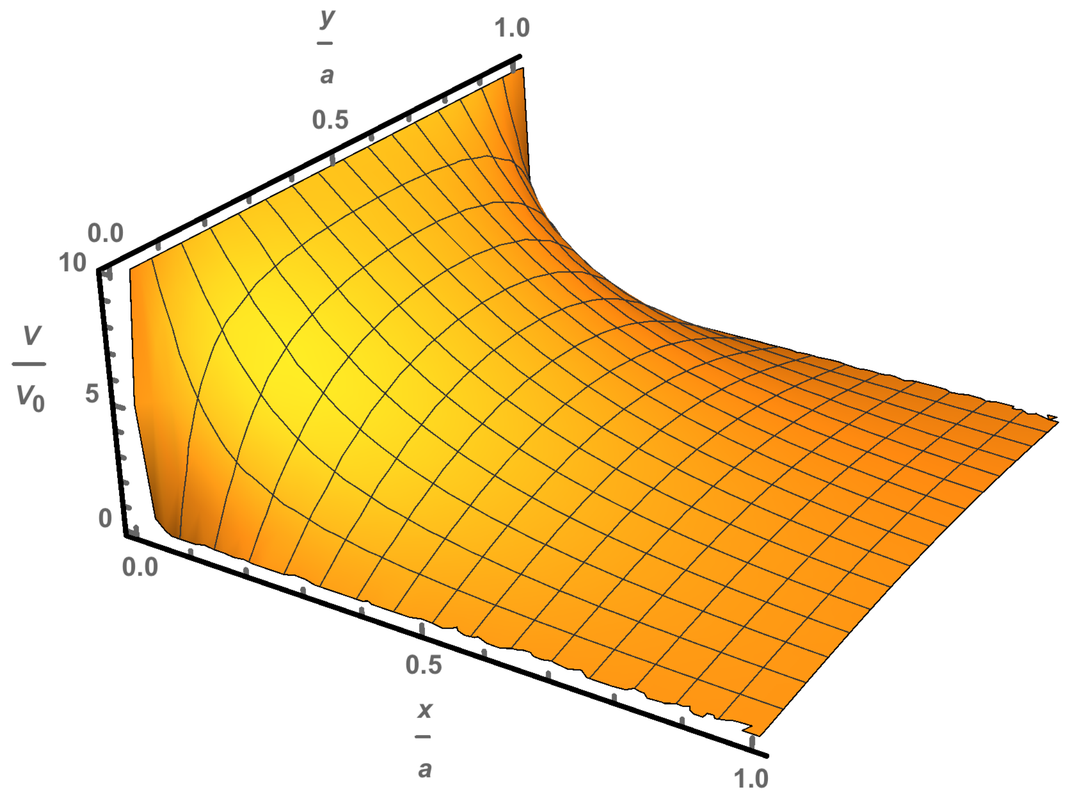
Example: Let $V_0(y) \equiv V_0 = \text{constant}$ for all y. Then, we have

$$\begin{aligned} C_m &= \frac{2}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right) V_0(y)dy \\ &= \frac{2V_0}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right) dy \\ &= \frac{2V_0}{n\pi}(1-\cos n\pi) \\ &= \begin{cases} \frac{4V_0}{n\pi} & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

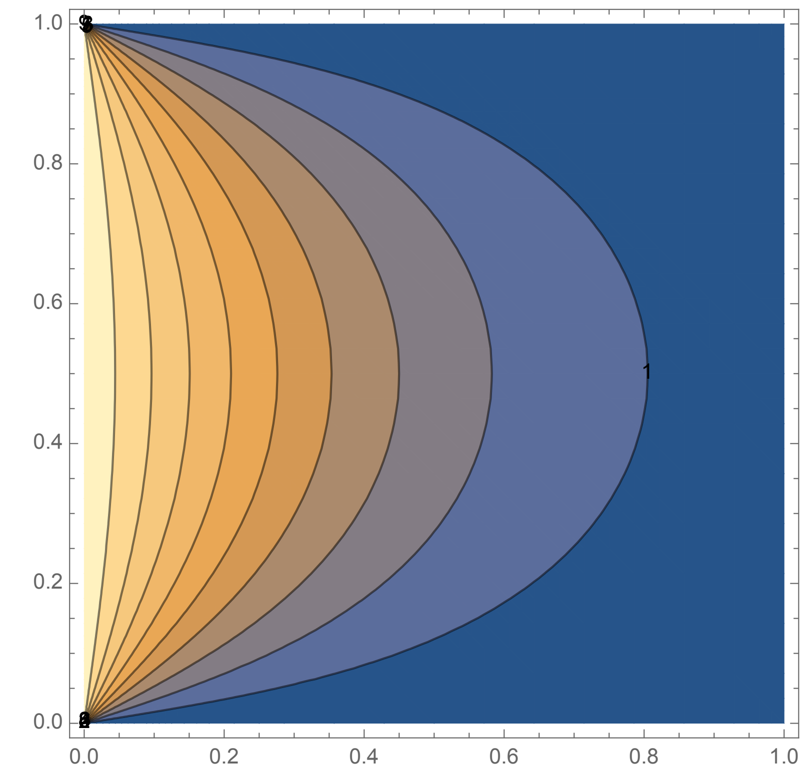
Therefore the final solution: $V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right).$

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\cdots} \frac{1}{n} e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right).$$

A plot of the Potential



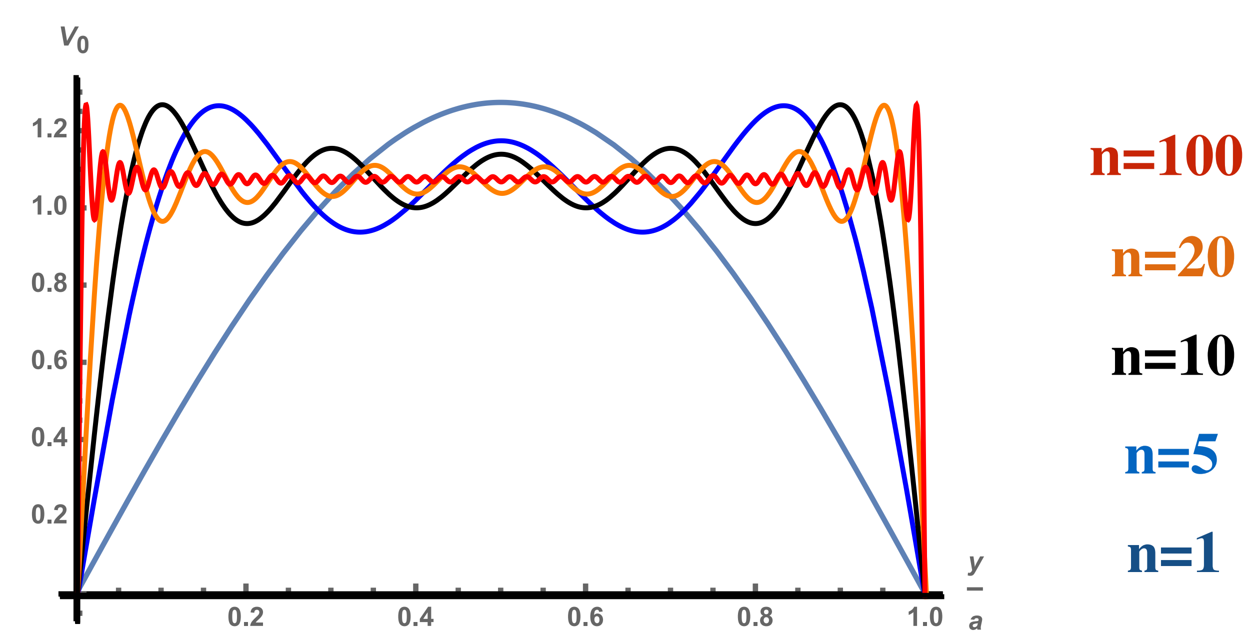
Plot of the potential



Equipotentials

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} e^{-n\pi x/a}\sin\left(\frac{n\pi y}{a}\right).$$

How the first few terms in the Fourier series combine to make a better and better approximation to V_0.



Example

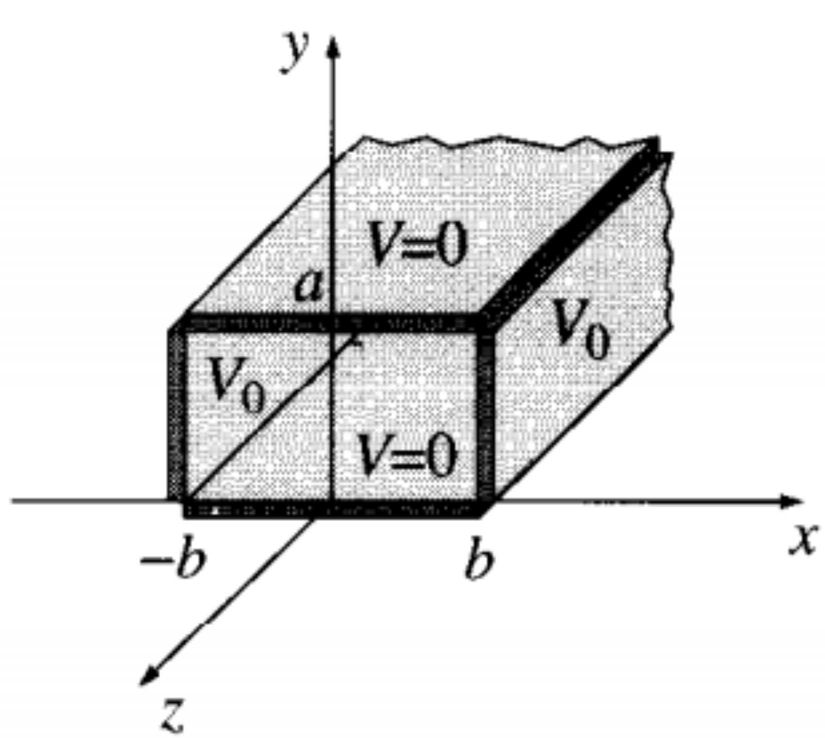
Two infinitely long grounded metal plates, again at $y = 0$ and $y = a$, are connected at $x = \pm b$ by metal strips maintained at a constant potential V_0, as shown in Fig. 3.20 (a thin layer of insulation at each corner prevents them from shorting out). Find the potential inside the resulting rectangular pipe.

Solution: Once again, the configuration is independent of z. Our problem is to solve Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

subject to the boundary conditions

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = V_0 \text{ when } x = b, \\ \text{(iv)} & V = V_0 \text{ when } x = -b. \end{array}\right\}$$



$$V(x, y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky).$$

This time, however, we cannot set $A = 0$; the region in question does not extend to $x = \infty$, so e^{kx} is perfectly acceptable. On the other hand, the situation is *symmetric* with respect to x, so $V(-x, y) = V(x, y)$, and it follows that $A = B$. Using

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = V_0 \text{ when } x = b, \\ \text{(iv)} & V = V_0 \text{ when } x = -b. \end{array}\right\}$$

$$e^{kx} + e^{-kx} = 2 \cosh kx,$$

and absorbing $2A$ into C and D, we have

$$V(x, y) = \cosh kx\, (C \sin ky + D \cos ky).$$

Boundary conditions (i) and (ii) require, as before, that $D = 0$ and $k = n\pi/a$, so

$$V(x, y) = C \cosh(n\pi x/a)\ \sin(n\pi y/a).$$

Because $V(x, y)$ is even in x, it will automatically meet condition (iv) if it fits (iii). It remains, therefore, to construct the general linear combination,

$$V(x, y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi x/a)\ \sin(n\pi y/a),$$

and pick the coefficients C_n in such a way as to satisfy condition (iii):

$$V(b, y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi b/a)\ \sin(n\pi y/a) = V_0.$$

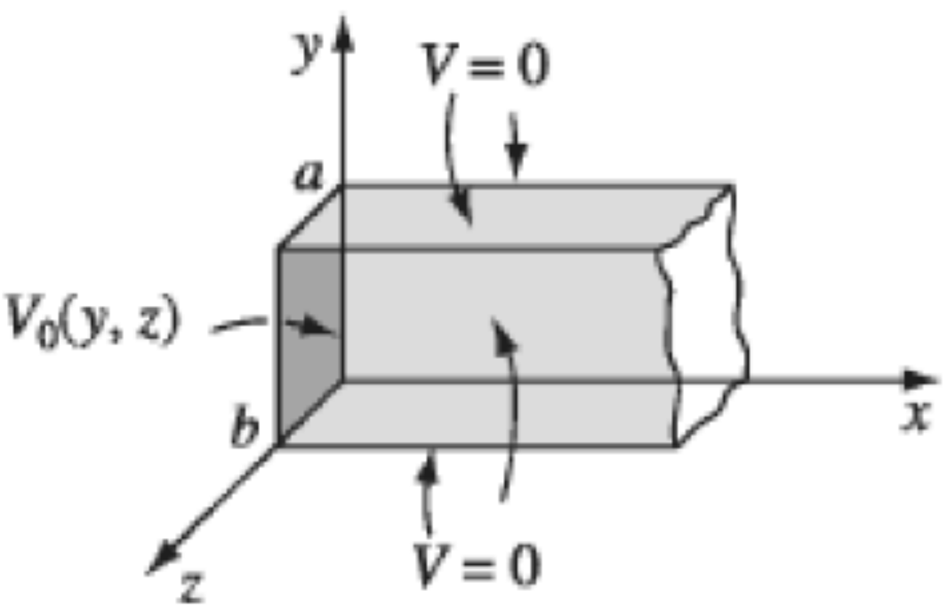
$$C_n \cosh(n\pi b/a) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4V_0}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

Conclusion: The potential in this case is given by

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} \frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)} \sin(n\pi y/a).$$

3D Laplace's Equation in Cartesian System

Example An infinitely long rectangular metal pipe (sides a and b) is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0(y, z)$. Find the potential inside the pipe.



(i) $V = 0$ when $y = 0$,
(ii) $V = 0$ when $y = a$,
(iii) $V = 0$ when $z = 0$,
(iv) $V = 0$ when $z = b$,
(v) $V \to 0$ as $x \to \infty$,
(vi) $V = V_0(y, z)$ when $x = 0$.

BC

This is a genuinely three-dimensional problem, $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(x, y, z) = X(x)Y(y)Z(z) \quad \Rightarrow \quad \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0.$$

Separation of Variables & Boundary Conditions

It follows that

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2, \quad \frac{1}{Z}\frac{d^2Z}{dz^2} = C_3, \quad \text{with } C_1 + C_2 + C_3 = 0.$$

Setting $C_2 = -k^2$ and $C_3 = -l^2$, we have $C_1 = k^2 + l^2$,

3 ODEs: $$\frac{d^2X}{dx^2} = (k^2 + l^2)X, \quad \frac{d^2Y}{dy^2} = -k^2Y, \quad \frac{d^2Z}{dz^2} = -l^2Z.$$

$$\Rightarrow \quad \begin{aligned} X(x) &= Ae^{\sqrt{k^2+l^2}\,x} + Be^{-\sqrt{k^2+l^2}\,x}, \\ Y(y) &= C\sin ky + D\cos ky, \\ Z(z) &= E\sin lz + F\cos lz. \end{aligned}$$

Boundary condition (v) implies $A = 0$, (i) gives $D = 0$, and (iii) yields $F = 0$, whereas (ii) and (iv) require that $k = n\pi/a$ and $l = m\pi/b$, where n and m are positive integers. Combining the remaining constants, we are left with

$$V(x, y, z) = Ce^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x} \sin(n\pi y/a) \sin(m\pi z/b).$$

Use of Fourier Trick

$$V(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,m}e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin(n\pi y/a)\,\sin(m\pi z/b)$$

B.C. (vi) :

$$V(0,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,m}\sin(n\pi y/a)\,\sin(m\pi z/b)=V_0(y,z)$$

Use Fourier Trick: multiply by $\sin(n'\pi y/a)\,\sin(m'\pi z/b)$,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,m}\int_0^a\sin(n\pi y/a)\,\sin(n'\pi y/a)\,dy\int_0^b\sin(m\pi z/b)\,\sin(m'\pi z/b)\,dz$$

$$=\int_0^a\int_0^b V_0(y,z)\sin(n'\pi y/a)\,\sin(m'\pi z/b)\,dy\,dz.$$

$$C_{n,m}=\frac{4}{ab}\int_0^a\int_0^b V_0(y,z)\sin(n\pi y/a)\,\sin(m\pi z/b)\,dy\,dz.$$

Final Solution for $V_0(y,z) = const.$

For instance, if the end of the tube is a conductor at *constant* potential $V_0 = V_0(y,z)$

$$C_{n,m} = \frac{4V_0}{ab}\int_0^a \sin(n\pi y/a)\,dy \int_0^b \sin(m\pi z/b)\,dz$$

$$= \begin{cases} 0, & \text{if } n \text{ or } m \text{ is even,} \\ \dfrac{16V_0}{\pi^2 nm}, & \text{if } n \text{ and } m \text{ are odd.} \end{cases}$$

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n,m=1,3,5\ldots}^{\infty}\frac{1}{nm}e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin(n\pi y/a)\,\sin(m\pi z/b)$$