An important point to note…

- Note that $\vec{\nabla}.\vec{D} = \rho_f$ just looks like Gauss's law, only the total charge density ρ has been replaced by free charge density ρ_f.

- Here $\vec{D}$ is the electric displacement vector $\epsilon_0 \vec{E} + \vec{P}$. But **do not** conclude that $\vec{D}$ is just like $\vec{E}$!

- In particular, there is no Coulomb's law for $\vec{D}$: $\vec{D}(\vec{r}) \neq \frac{1}{4\pi} \int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho_f(\vec{r}\,') d\tau'$

- Curl of the electric field is **always** zero! But curl of $\vec{D}$ is not always zero.

$$\vec{\nabla} \times \vec{D} = \epsilon_0 (\vec{\nabla} \times \vec{E}) + (\vec{\nabla} \times P) = \vec{\nabla} \times \vec{P}$$

  and there is no reason in general to suppose that curl of $\vec{P}$ is zero. Sometimes it may, but not in general.

- Because $\vec{\nabla} \times \vec{D} \neq 0$, moreover, $\vec{D}$ can not be expressed as gradient of a scalar - there is no potential for $\vec{D}$.

Boundary conditions

• The electrostatic boundary conditions can be represented in terms of $\vec{D}$. We already know:

$$\oint \vec{D}.d\vec{a} = Q_{\mathrm{f_{enc}}} \quad \text{and} \quad \vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$$

Talks about discontinuity in component perpendicular to an interface.

Talks about discontinuity in parallel component along the interface.

$$D^{\perp}_{\mathrm{above}} - D^{\perp}_{\mathrm{below}} = \sigma_f$$

$$D^{\parallel}_{\mathrm{above}} - D^{\parallel}_{\mathrm{below}} = P^{\parallel}_{\mathrm{above}} - P^{\parallel}_{\mathrm{below}}$$

• In presence of dielectrics, these are sometimes more useful that the corresponding boundary conditions on $\vec{E}$:

$$E^{\perp}_{\mathrm{above}} - E^{\perp}_{\mathrm{below}} = \frac{\sigma}{\epsilon_0} \quad \text{and} \quad E^{\parallel}_{\mathrm{above}} - E^{\parallel}_{\mathrm{below}} = 0$$

Linear Dielectric

What causes polarization in a material ? **Electric field that lines up atomic or molecular dipoles**

- For many substances, polarization is linearly proportional to electric field → **Linear Dielectric**

(provided ***E*** is not too strong)

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

χ_e → Electric susceptibility

- Susceptibility of a material depends on the microscopic structure
- dimensionless quantity

- Note: **The electric field above is not only the external field, but also includes the contribution due to polarization.** We can't compute P directly from above eqn.

- Easier way to parametrise it, is to identify the electric displacement for which

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1 + \chi_e) \vec{E} = \epsilon \vec{E}$$

$$\epsilon = \epsilon_0 (1 + \chi_e)$$ → Permittivity of the material

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e$$ → Relative permittivity or Dielectric constant

Polarisation of crystal

Although, material is still linear dielectric, polarisation of some materials are different in different direction.

Recall, we had $$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

A crystal is generally easier to polarize in some directions than in others and the above relation changes to a more general linear relation of the form:

$$P_x = \epsilon_0(\chi_{xx}E_x + \chi_{xy}E_y + \chi_{xz}E_z)$$
$$P_y = \epsilon_0(\chi_{yx}E_x + \chi_{yy}E_y + \chi_{yz}E_z)$$
$$P_z = \epsilon_0(\chi_{zx}E_x + \chi_{zy}E_y + \chi_{zz}E_z)$$

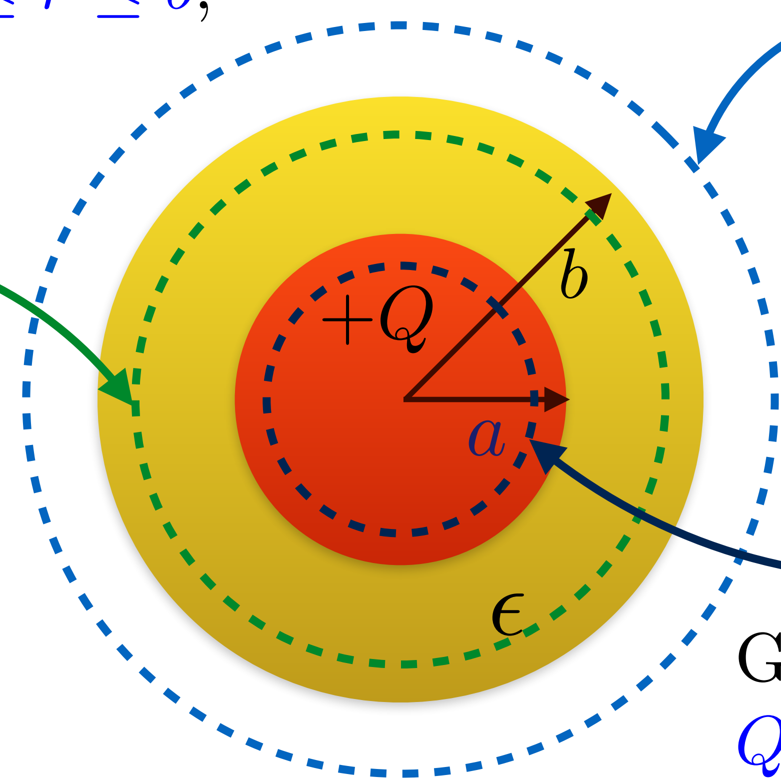
χ_{ij} forms Susceptibility tensor

Example

A metal sphere of radius a carries a charge $+Q$. It is surrounded out to radius b, by a linear dielectric material of permittivity ϵ . What is the potential at the centre (with respect to infinity)? What are the bound charges ?

Gaussian surface for which $r > b$, $Q_{\text{encl}} = Q$

Gaussian surface for which $a \leq r \leq b$, $Q_{\text{encl}} = Q$



Gaussian surface for which $r < a$, $Q_{\text{encl}} = 0$

Recall Gauss's law for dielectrics:

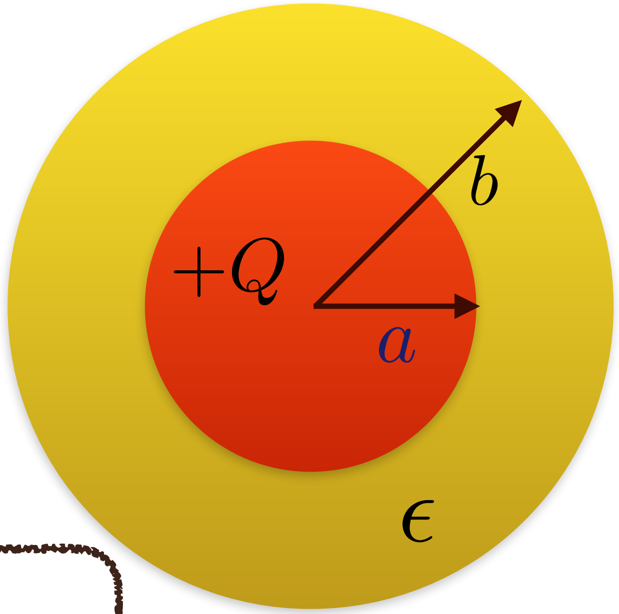
$$\oint \vec{D}.d\vec{a} = Q_{\text{f}_{\text{enc}}}$$

$$|\vec{D}|4\pi r^2 = Q$$

- Therefore the displacement: $\boxed{\vec{D} = \frac{Q}{4\pi r^2}\hat{r}}$ for all points $r > a$
- Inside the metal sphere of course: $\vec{E} = \vec{D} = \vec{P} = 0$
- Hence we have: $\vec{E} = 0$ for $r < a$

Example

A metal sphere of radius a carries a charge $+Q$. It is surrounded out to radius b, by a linear dielectric material of permittivity ϵ . What is the potential at the centre (with respect to infinity)? What are the bound charges ?



- Once we know $\vec{D}$, we can calculate electric field, since $\vec{D} = \epsilon\vec{E}$

$$\vec{E} = \begin{cases} \frac{Q}{4\pi\epsilon r^2}\hat{r} & \text{for} \quad a < r < b \\ \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & \text{for} \quad r > b. \end{cases}$$

- Hence, potential at centre

$$V = -\int_{\infty}^{0} \vec{E}.d\vec{l} = -\int_{\infty}^{b} \frac{Q}{4\pi\epsilon_0 r^2} dr - \int_{b}^{a} \frac{Q}{4\pi\epsilon r^2} dr - \int_{a}^{0} 0 \; dr$$

$$= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right)$$

Note that we did not have to calculate the polarization or bound charges explicitly for calculation the potential, although we can do so:

Polarisation

$$\vec{P} = \epsilon_0\chi_e\vec{E} = \frac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\hat{r}$$

Bound charges

$$\rho_b = -\vec{\nabla}.\vec{P} = 0$$

$$\sigma_b = \vec{P}.\hat{n} = \begin{cases} \frac{\epsilon_0\chi_e Q}{4\pi\epsilon b^2} & \text{outer surface} \\ \frac{-\epsilon_0\chi_e Q}{4\pi\epsilon a^2} & \text{inner surface.} \end{cases}$$

Curl of Polarisation in linear dielectric

We might now say, that as polarisation and displacement are proportional to field in linear media, the curl of them should vanish $\vec{\nabla}\times\vec{P}=0, \quad \vec{\nabla}\times\vec{D}=0$

If the medium is homogeneously filled with dielectric of one kind, then, indeed so, otherwise not. $\vec{\nabla}\times\vec{P}=0, \quad \vec{\nabla}\times\vec{D}=0, \quad \vec{\nabla}.\vec{D}=\rho_f$

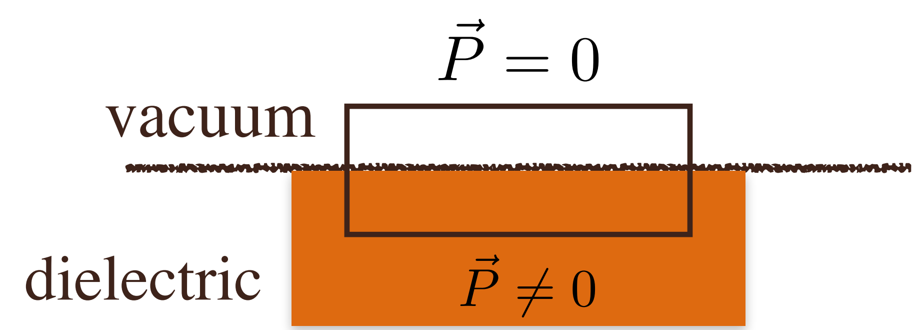
So, D can be found out from the free charges as if the dielectric is not there $\vec{D}=\epsilon_0\vec{E}_{vac}$

The field, the same ρ_f would produce in absence of dielectric.

Using $\vec{D}=\epsilon\vec{E}$ and $\epsilon_r\equiv 1+\chi_e=\frac{\epsilon}{\epsilon_0}$: $\vec{E}=\frac{1}{\epsilon}\vec{D}=\frac{1}{\epsilon_r}\vec{E}_{vac}$

Hence, when the space is filled with homogeneous linear dielectric, the field is reduced by $\epsilon_r=\frac{\epsilon}{\epsilon_0}$

- Consider, the interface of two different medium, eg: dielectric with vacuum:



$$\oint \vec{P}.d\vec{l}\neq 0 \longrightarrow \vec{\nabla}\times\vec{P}\neq 0$$

$$\longrightarrow \vec{\nabla}\times\vec{D}=\vec{\nabla}\times(\epsilon_0\vec{E}+\vec{P})\neq 0$$

What does a free charge do in dielectric medium ?

Suppose a free charge is embedded in a large dielectric.

It produces a field: $\vec{E} = \frac{1}{4\pi\epsilon}\frac{q}{r^2}\hat{r}$ $\quad(\epsilon \text{ not } \epsilon_0)$

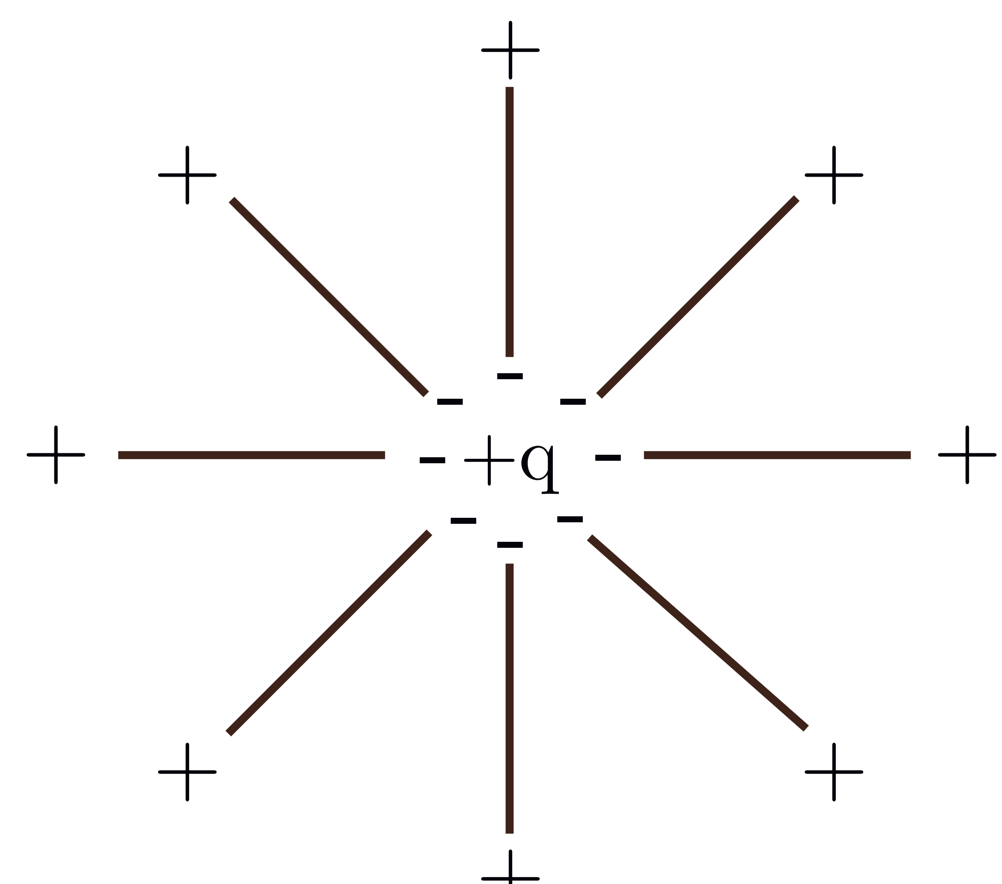
It can be simply obtained as follows:

$$\vec{E} = \frac{1}{\epsilon}\vec{D} = \frac{1}{\epsilon_r}\vec{E}_{vac}$$

with $\vec{E}_{vac} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$ and $\epsilon_r = \frac{\epsilon}{\epsilon_0}$

Physically: Polarisation shields the charge by surrounding it with bound charges of opposite signs

This causes reduction in the field in the dielectric material

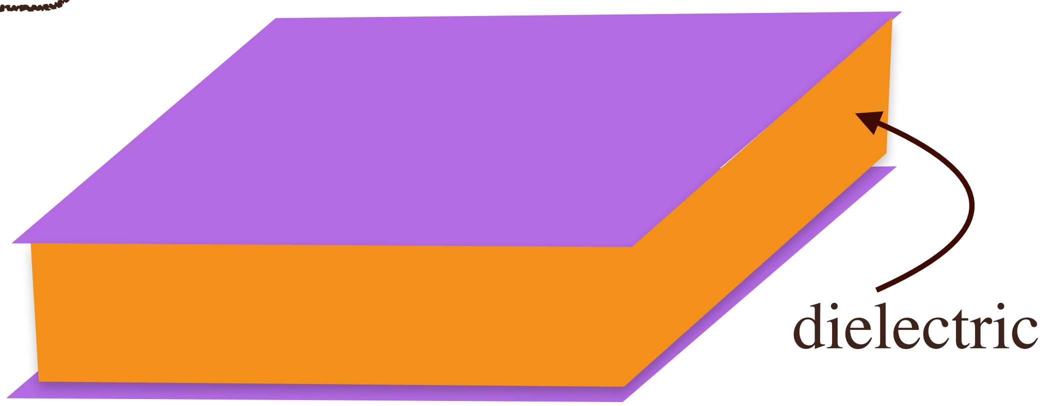


Capacitor filled with linear dielectric material : revisited

We can simply use the relation of electric field inside the dielectric material compared to the case of vacuum

$$\vec{E} = \frac{1}{\epsilon}\vec{D} = \frac{1}{\epsilon_r}\vec{E}_{vac}$$

$$\longrightarrow \quad V = \frac{V_{vac}}{\epsilon_r}$$



$$C = \frac{Q}{V} = \epsilon_r \frac{Q}{V_{vac}} = \epsilon_r C_{vac}$$

Capacitance increases by the factor of dielectric constant of the material

$$\epsilon_r = \frac{\epsilon}{\epsilon_0}$$