Physics II: Electromagnetism

PH 102

Lecture 2

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Vector Integral Calculus

Line, Surface and Volume integrals

Line Integration

Line Integrals

Extension of idea of integration of one variable

$$\int_a^b f(x)dx$$

to scalar and vector fields on any paths.

Naturally the question arises : "how to define paths?"

For that we need to review a little bit on parametric equations and curves.

Most familiar example: Equation of trajectory of a particle

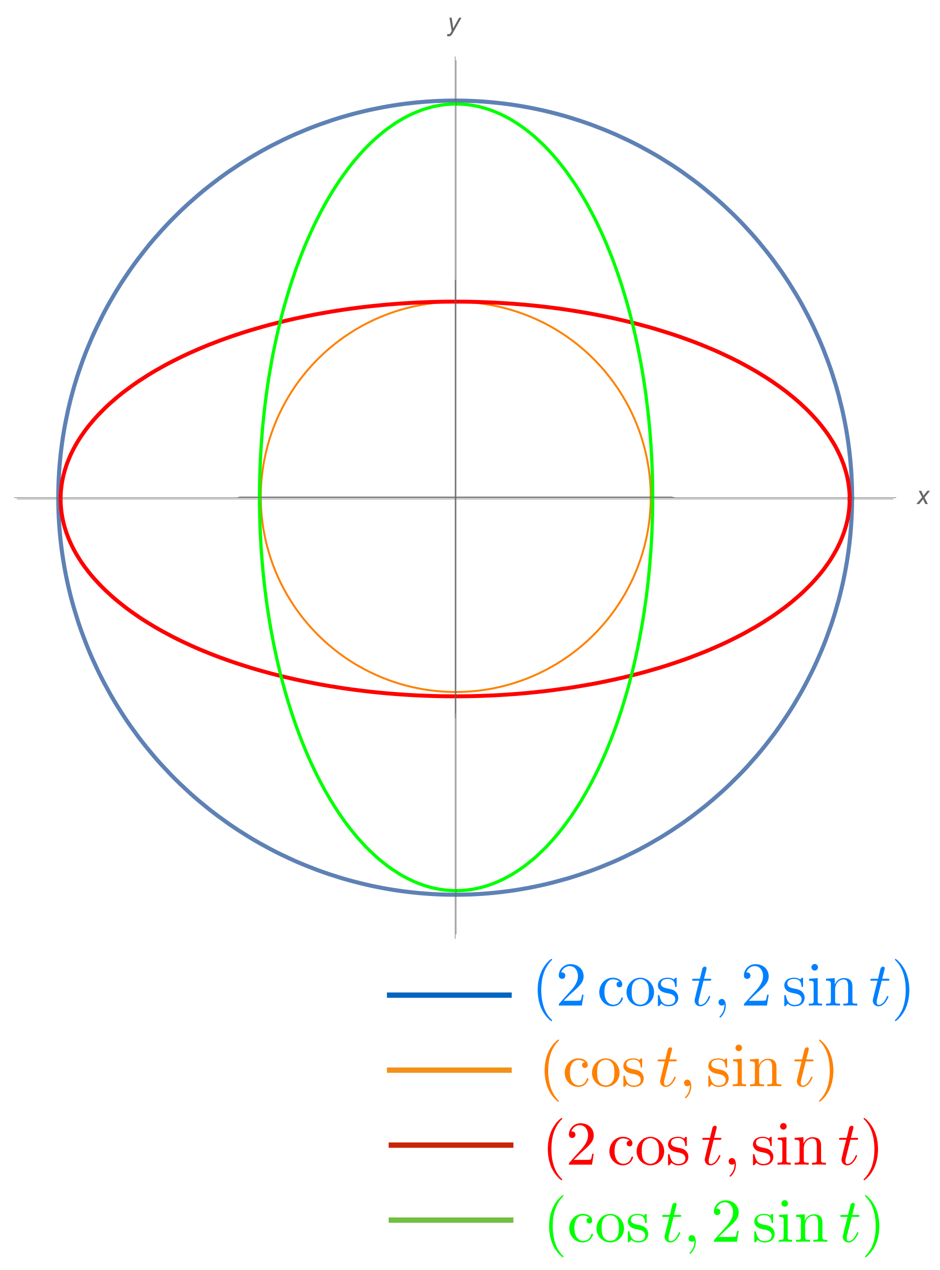
1D: $x = x(t)$

2D: $x = x(t); y = y(t)$

3D: $x = x(t); y = y(t); z = z(t)$

How to describe paths: Parametric equations and curves

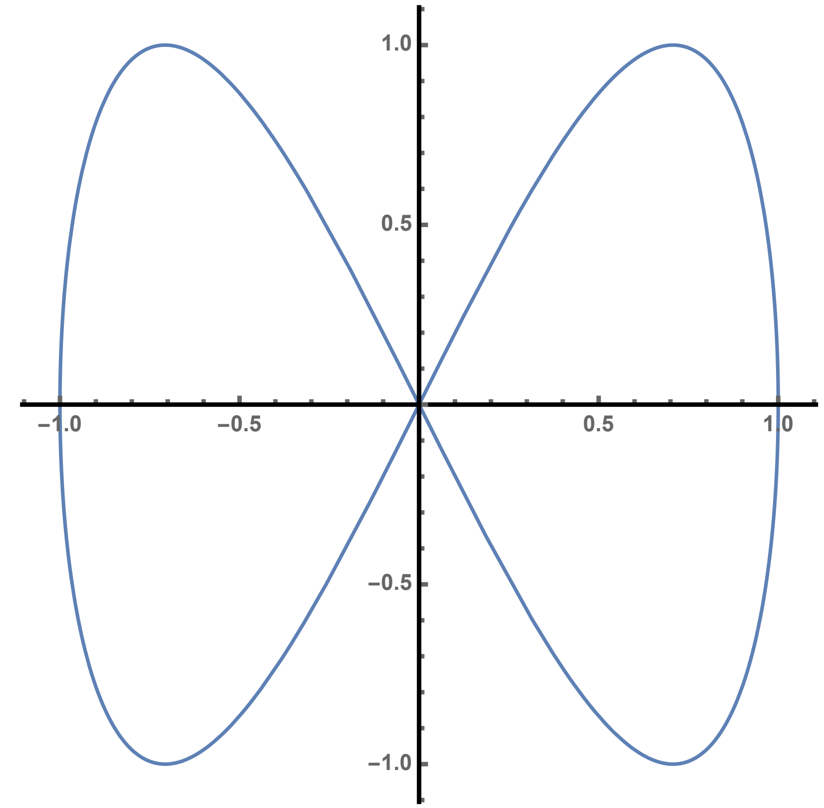
Curve	Parametric equations
$x^2 + y^2 = R^2$	$\begin{aligned} x &= R\cos t, \\ y &= R\sin t \end{aligned} \quad 0 \le t \le 2\pi$
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$\begin{aligned} x &= a\cos t, \\ y &= b\sin t \end{aligned} \quad 0 \le t \le 2\pi$
$y = f(x)$	$\begin{aligned} x &= t, \\ y &= f(t) \end{aligned}$
Line segment from (x_1, y_1, z_1) to (x_2, y_2, z_2).	$\begin{aligned} x &= (1-t)x_1 + tx_2 \\ y &= (1-t)y_1 + ty_2\,, \; 0 \le t \le 1 \\ z &= (1-t)z_1 + tz_2 \end{aligned}$



More complicated examples

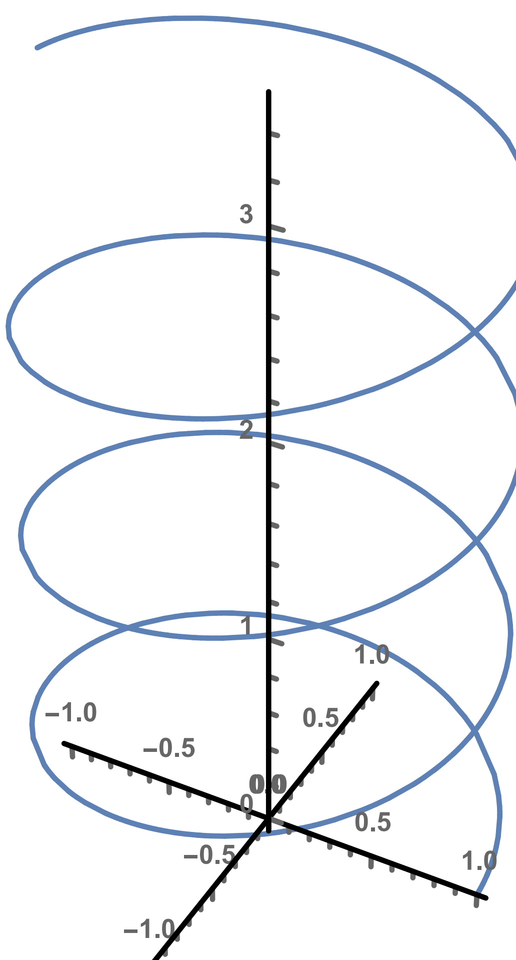
The figure of eight curve

$$\left.\begin{array}{rcl} x(t) & = & \sin t, \\ y(t) & = & \sin 2t \end{array}\right\} \quad 0 \leq t \leq 2\pi$$

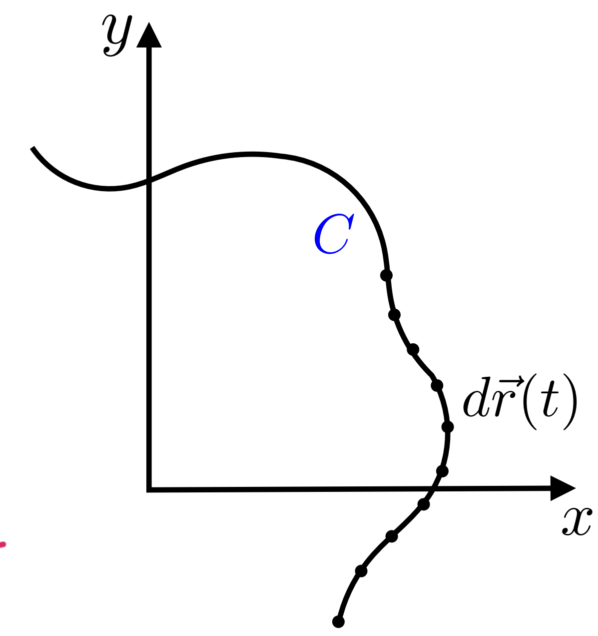


Helix

$$\left.\begin{array}{rcl} x(t) & = & \sin t, \\ y(t) & = & \cos t, \\ z(t) & = & t/2\pi \end{array}\right\} \quad 0 \leq t \leq 7\pi$$



Line integrals: Scalar field



Now, divide the path into small segments: $d\vec{r}$.

$$d\vec{r}(t) = dx(t)\ \hat{x} + dy(t)\ \hat{y} = \left(\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y}\right) dt$$

$$= \vec{r}\,'(t)dt$$

Length of the segment:

$$|d\vec{r}| = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2} dt = |\vec{r}\,'(t)|dt$$

Line integral of a scalar field f over a curve C (whose parametric representation is given by the path $\vec{r}(t)$) is

$$\int_C f dr = \int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|dt.$$

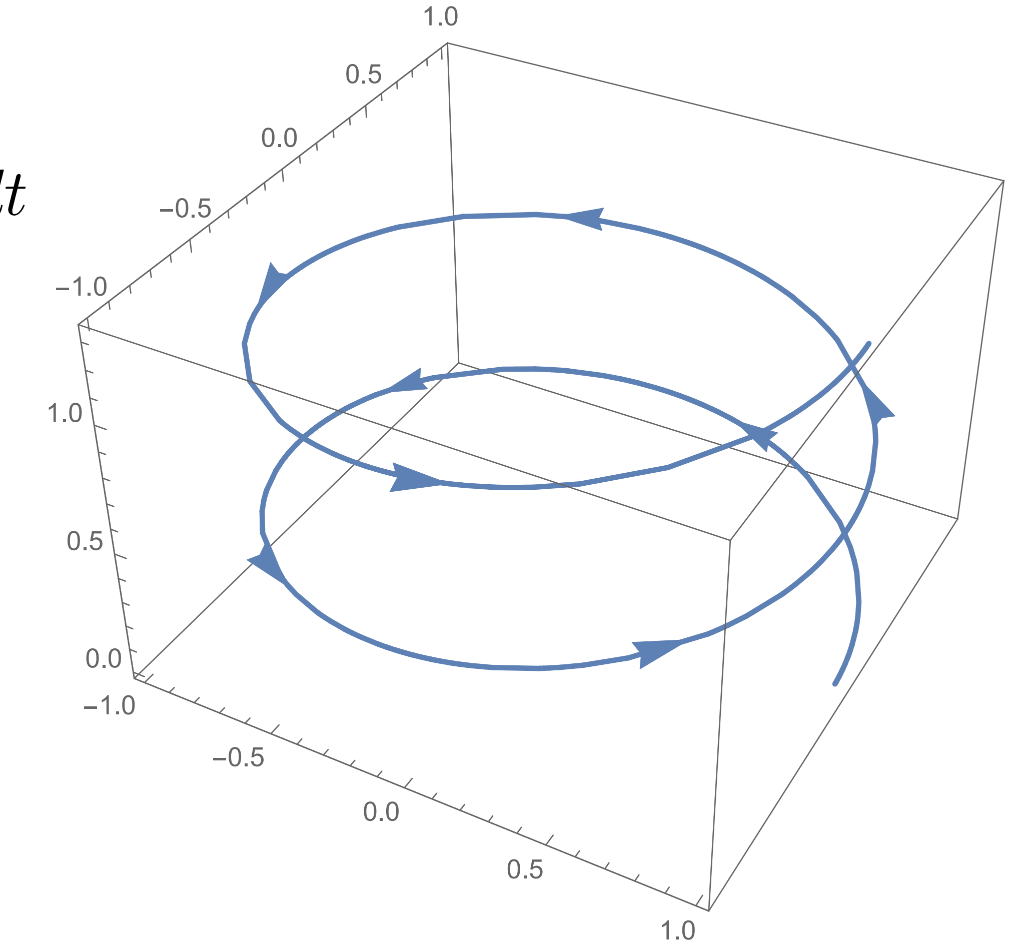
Generalisation for a function of three variables is straightforward : only change -

$$\begin{aligned} d\vec{r}(t) &= dx(t)\ \hat{x} + dy(t)\ \hat{y} + dz(t)\ \hat{z} \\ |\vec{r}\,'(t)| &= [(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2]^{1/2} \end{aligned}$$

Example

Evaluate $\int_C xyz\ dr$ where C is the helix given by $\vec{r}(t) = (\cos t,\ \sin t,\ 3t)$, $0 \le t \le 4\pi$.

$$
\begin{aligned}
\int_C xyz\ dr &= \int_0^{4\pi} \overbrace{3t\cos t\sin t}^{f(\vec{r}(t))}\ \overbrace{\sqrt{\sin^2 t + \cos^2 t + 9}}^{|\vec{r}\,'(t)|}\, dt \\
&= \int_0^{4\pi} 3t\left(\frac{1}{2}\sin 2t\right)\sqrt{1+9}dt \\
&= \frac{3\sqrt{10}}{2}\int_0^{4\pi} t\sin 2t dt \\
&= \frac{3\sqrt{10}}{2}\left(\frac{1}{4}\sin 2t - \frac{t}{2}\cos 2t\right)\Bigg|_0^{4\pi} \\
&= -3\sqrt{10}\pi
\end{aligned}
$$



Example 2 : If $\phi = 2xyz^2$, find $\boxed{\int_C \phi \, d\vec{r}}$

where C is the curve

$$x = t^2, \quad y = 2t, \quad z = t^3$$

$$(0 \leq t \leq 1).$$

Along C : $\phi = 2xyz^2 = 2 \cdot t^2 \cdot 2t \cdot t^6$

$$= 4t^9.$$

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

$$= t^2\,\hat{x} + 2t\,\hat{y} + t^3\,\hat{z}$$

$$\therefore \quad d\vec{r} = \left(2t\,\hat{x} + 2\,\hat{y} + 3t^2\,\hat{z}\right) dt$$

$$\therefore \int_C \phi \, d\vec{r} = \int_0^1 (4+t^9)\left(2t\,\hat{x} + 2\,\hat{y} + 3t^2\,\hat{z}\right) dt$$

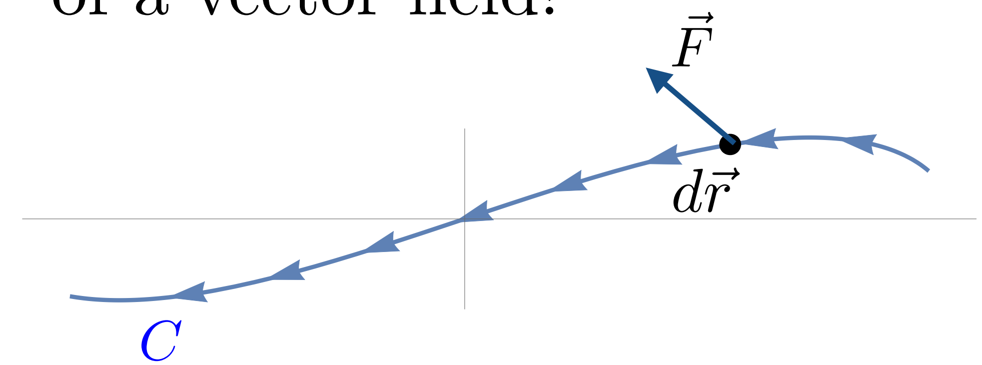
$$= \hat{x}\int_0^1 8t^{10}\,dt + \hat{y}\int_0^1 8t^9\,dt + \hat{z}\int_0^1 12t^{11}\,dt$$

$$= \frac{8}{11}\,\hat{x} + \frac{4}{5}\,\hat{y} + \hat{z}$$

Line Integrals: Vector fields

Recall: while calculating the work done by a force along the direction of the motion of a particle, you basically did "line integral" of a vector field!

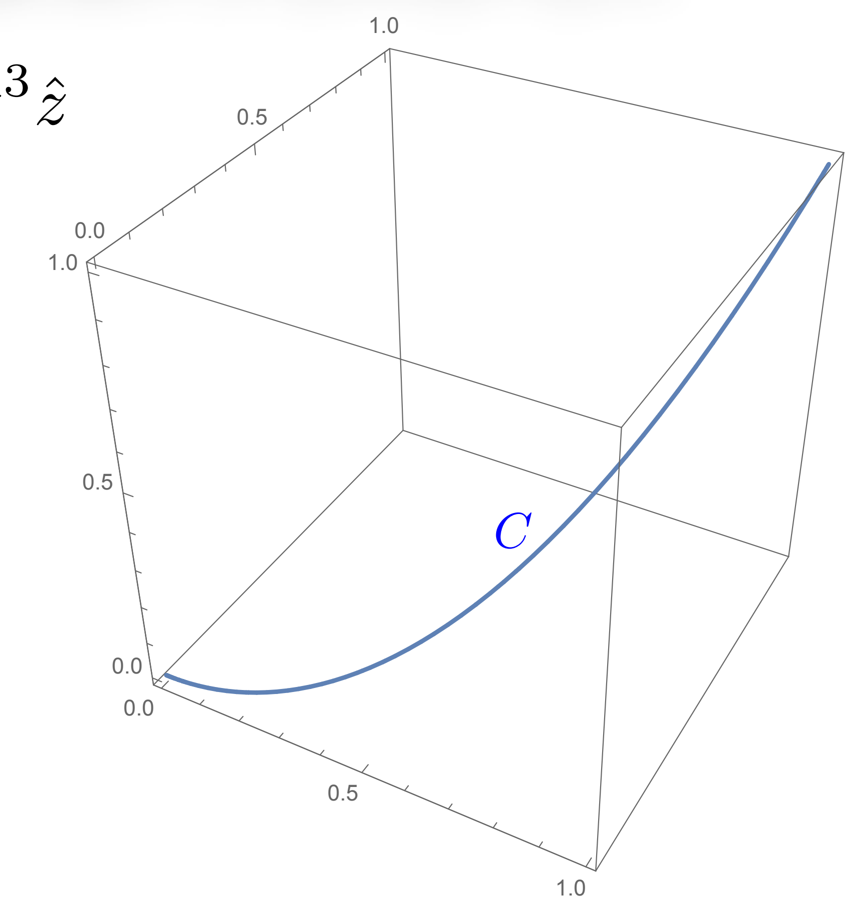
$$\int_C \vec{F}.d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)).\vec{r}\,'(t)\ dt$$



$\vec{r}(t)$ in the range $a \le t \le b$ is the parametric representation of path C.

Ex: A force field is given by $\vec{F}(x,y,z) = 8x^2yz\ \hat{x} + 5z\ \hat{y} - 4xy\ \hat{z}$. Find the work done in moving a particle along a curve parametrised by (t, t^2, t^3); $0 \le t \le 1$.

$$\vec{F}(\vec{r}(t)) = 8t^2(t^2)(t^3)\hat{x} + 5t^3\hat{y} - 4t(t^2)\hat{z} = 8t^7\hat{x} + 5t^3\hat{y} - 4t^3\hat{z}$$



Parametric path :

$$\begin{aligned} \vec{r}(t) &= t\hat{x} + t^2\hat{y} + t^3\hat{z} \\ \Longrightarrow \vec{r}\,'(t) &= \hat{x} + 2t\hat{y} + 3t^2\hat{z} \end{aligned}$$

Work done :

$$\begin{aligned} \int_C \vec{F}.d\vec{r} &= \int_0^1 \Big(8t^7 + 10t^4 - 12t^5\Big) dt \\ &= (t^8 + 2t^5 - 2t^6)\Big|_0^1 \\ &= 1 \end{aligned}$$

Example 3

If $\mathbf{A} = (3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ from (0,0,0) to (1,1,1) along the following paths C:

(a) $x = t,\ y = t^2,\ z = t^3$.

(b) the straight lines from (0,0,0) to (1,0,0), then to (1,1,0), and then to (1,1,1).

(c) the straight line joining (0,0,0) and (1,1,1).

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_C \left[(3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\mathbf{k}\right]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

(a) If $x = t$, $y = t^2$, $z = t^3$, points (0,0,0) and (1,1,1) correspond to $t = 0$ and $t = 1$ respectively. Then

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (3t^2+6t^2)\,dt - 14(t^2)(t^3)\,d(t^2) + 20(t)(t^3)^2\,d(t^3)$$

$$= \int_{t=0}^{1} 9t^2\,dt - 28t^6\,dt + 60t^9\,dt$$

$$= \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\,dt = 3t^3 - 4t^7 + 6t^{10}\Big|_0^1 = 5$$

Another Method.

Along C, $\mathbf{A} = 9t^2\mathbf{i} - 14t^5\mathbf{j} + 20t^7\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ and $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\,dt$.

$$\text{Then} \quad \int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (9t^2\,\mathbf{i} - 14t^5\,\mathbf{j} + 20t^7\,\mathbf{k})\cdot(\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k})\,dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt = 5$$

(*b*) Along the straight line from $(0,0,0)$ to $(1,0,0)$ $y = 0$, $z = 0$, $dy = 0$, $dz = 0$ while x varies from 0 to 1. Then the integral over this part of the path is

$$\int_{x=0}^{1} \big(3x^2 + 6(0)\big)\,dx - 14(0)(0)(0) + 20x(0)^2(0) = \int_{x=0}^{1} 3x^2\,dx = x^3\Big|_0^1 = 1$$

Along the straight line from $(1,0,0)$ to $(1,1,0)$ $x = 1$, $z = 0$, $dx = 0$, $dz = 0$ while y varies from 0 to 1. Then the integral over this part of the path is

$$\int_{y=0}^{1} (3(1)^2 + 6y)\,0 - 14y(0)\,dy + 20(1)(0)^2\,0 = 0$$

Along the straight line from (1,1,0) to (1,1,1) $x = 1$, $y = 1$, $dx = 0$, $dy = 0$ while z varies from 0 to 1. Then the integral over this part of the path is

$$\int_{z=0}^{1} \left(3(1)^2+6(1)\right)0 \;-\; 14(1)\,z(0) \;+\; 20(1)\,z^2\,dz \;=\; \int_{z=0}^{1} 20z^2\,dz \;=\; \frac{20z^3}{3}\Big|_0^1 \;=\; \frac{20}{3}$$

Adding,
$$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; 1 + 0 + \frac{20}{3} \;=\; \frac{23}{3}$$

(c) The straight line joining (0,0,0) and (1,1,1) is given in parametric form by $x = t$, $y = t$, $z = t$. Then

$$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; \int_{t=0}^{1} (3t^2+6t)\,dt \;-\; 14(t)(t)\,dt \;+\; 20(t)(t)^2\,dt$$

$$=\; \int_{t=0}^{1} (3t^2+6t-14t^2+20t^3)\,dt \;=\; \int_{t=0}^{1} (6t-11t^2+20t^3)\,dt \;=\; \frac{13}{3}$$

Example 4

If $\mathbf{F} = 3xy\,\mathbf{i} - y^2\,\mathbf{j}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve in the xy plane, $y = 2x^2$, from $(0,0)$ to $(1,2)$.

Since the integration is performed in the xy plane $(z=0)$, we can take $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (3xy\,\mathbf{i} - y^2\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C 3xy\,dx - y^2\,dy$$

First Method. Let $x = t$ in $y = 2x^2$. Then the parametric equations of C are $x = t$, $y = 2t^2$. Points $(0,0)$ and $(1,2)$ correspond to $t = 0$ and $t = 1$ respectively. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t=0}^{1} 3(t)(2t^2)\,dt - (2t^2)^2\,d(2t^2) = \int_{t=0}^{1} (6t^3 - 16t^5)\,dt = -\frac{7}{6}$$

Second Method. Substitute $y = 2x^2$ directly, where x goes from 0 to 1. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{x=0}^{1} 3x(2x^2)\,dx - (2x^2)^2\,d(2x^2) = \int_{x=0}^{1} (6x^3 - 16x^5)\,dx = -\frac{7}{6}$$

Note that if the curve were traversed in the opposite sense, i.e. from $(1,2)$ to $(0,0)$, the value of the integral would have been $7/6$ instead of $-7/6$.

More example

A force field is given by $\kappa \hat{r}/|\vec{r}|^2$, where $\kappa > 0$ is a constant and $\vec{r}$ is the position vector. What is the work done in moving a particle along a curve $\vec{r}(t) = (\cos t, \sin t);\ 0 \le t \le 2\pi$.

Along the path, $|\vec{r}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1$ and $F(\vec{r}(t)) = \kappa\hat{r}/|\vec{r}|^2 = \kappa\vec{r}/|\vec{r}|^3$.

Note: path is a circle and it is closed (start (0) and end points (2π)) are same.

Hence, $\vec{r}\,'(t) = (-\sin t, \cos t)$. Therefore, the work done:

$$\int_0^{2\pi} \vec{F}(\vec{r}(t)).\vec{r}\,'(t)dt = \kappa \int_0^{2\pi} \frac{(\cos t, \sin t).(-\sin t, \cos t)}{|\vec{r}(t)|^3} \quad |\vec{r}(t)|^3 = 1$$

$$= \kappa \int_0^{2\pi} (-\cos t \sin t + \sin t \cos t)dt = 0$$

$\boxed{\text{Did you expect this? Why? Check that } \vec{\nabla} \times \kappa\vec{r}/|\vec{r}|^3 = 0}$

However, take another vector field $\vec{F} = (-y\hat{x} + x\hat{y})$ from the previous lecture.

with $\vec{r} = (\cos t, \sin t)$

$$\int_0^{2\pi} \vec{F}(\vec{r}(t)).\vec{r}\,'(t)dt = \int_0^{2\pi} (-\sin t, \cos t).(-\sin t, \cos t)dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)dt = 2\pi$$

Remember what the curl of this field was? It was $2\hat{z} \neq 0$

Conservative vector field

If for a vector field $\vec{F}(x, y, z)$, $\vec{\nabla} \times \vec{F} = 0$, then we have seen that $\vec{F} = \vec{\nabla}\phi$, where ϕ is a scalar field.

$$
\begin{aligned}
\text{Then,} \quad \int_a^b \vec{F}.d\vec{r} &= \int_a^b \vec{\nabla}\phi.d\vec{r} = \int_a^b \left(\hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}\right).(dx\hat{x} + dy\hat{y} + dz\hat{z}) \\
&= \int_a^b \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) \\
&= \int_a^b d\phi = \phi(b) - \phi(a)
\end{aligned}
$$

As $\phi = \phi(x, y, z)$

Fundamental theorem for gradients. Will discuss later in detail.

if a vector field is expressible as a gradient of a scalar function, then the line integral would depend only on end points and not depend on path. Such a field is called conservative.

Example:

(*a*) Show that $\mathbf{F} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$ is a conservative force field. (*b*) Find the scalar potential. (*c*) Find the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

(*a*) From Problem 11, a necessary and sufficient condition that a force will be conservative is that $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \mathbf{0}$.

$$\text{Now}\quad \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} = \mathbf{0}.$$

Thus $\mathbf{F}$ is a conservative force field.

(*b*) *First Method.*

By Problem 10, $\mathbf{F} = \nabla\phi$ or $\dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$. Then

$$(1)\ \frac{\partial\phi}{\partial x} = 2xy + z^3 \qquad (2)\ \frac{\partial\phi}{\partial y} = x^2 \qquad (3)\ \frac{\partial\phi}{\partial z} = 3xz^2$$

Integrating, we find from (*1*), (*2*) and (*3*) respectively,

$$\begin{aligned}
\phi &= x^2y + xz^3 + f(y,z) \\
\phi &= x^2y + g(x,z) \\
\phi &= xz^3 + h(x,y)
\end{aligned}$$

These agree if we choose $f(y,z) = 0$, $g(x,z) = xz^3$, $h(x,y) = x^2y$ so that $\phi = x^2y + xz^3$ to which may be added any constant.

Second Method.

Since $\mathbf{F}$ is conservative, $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C joining (x_1, y_1, z_1) and (x,y,z). Using the method of Problem 11(*b*),

$$\begin{aligned}
\phi(x,y,z) &= \int_{x_1}^{x} (2xy_1 + z_1^3)\,dx + \int_{y_1}^{y} x^2\,dy + \int_{z_1}^{z} 3xz^2\,dz \\
&= (x^2y_1 + xz_1^3)\Big|_{x_1}^{x} + x^2y\Big|_{y_1}^{y} + xz^3\Big|_{z_1}^{z} \\
&= x^2y_1 + xz_1^3 - x_1^2y_1 - x_1z_1^3 + x^2y - x^2y_1 + xz^3 - xz_1^3 \\
&= x^2y + xz^3 - x_1^2y_1 - x_1z_1^3 = x^2y + xz^3 + \text{constant}
\end{aligned}$$

Third Method. $\mathbf{F}\cdot d\mathbf{r} = \nabla\phi\cdot d\mathbf{r} = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$

Then
$$\begin{aligned} d\phi = \mathbf{F}\cdot d\mathbf{r} &= (2xy+z^3)\,dx + x^2\,dy + 3xz^2\,dz \\ &= (2xy\,dx + x^2\,dy) + (z^3\,dx + 3xz^2\,dz) \\ &= d(x^2y) + d(xz^3) = d(x^2y + xz^3) \end{aligned}$$

and $\phi = x^2y + xz^3 + \text{constant}$.

(*c*) Work done $= \int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r}$

$$= \int_{P_1}^{P_2} (2xy+z^3)\,dx + x^2\,dy + 3xz^2\,dz$$

$$= \int_{P_1}^{P_2} d(x^2y+xz^3) = x^2y + xz^3\Big|_{P_1}^{P_2} = x^2y + xz^3\Big|_{(1,-2,1)}^{(3,1,4)} = 202$$

Another Method.

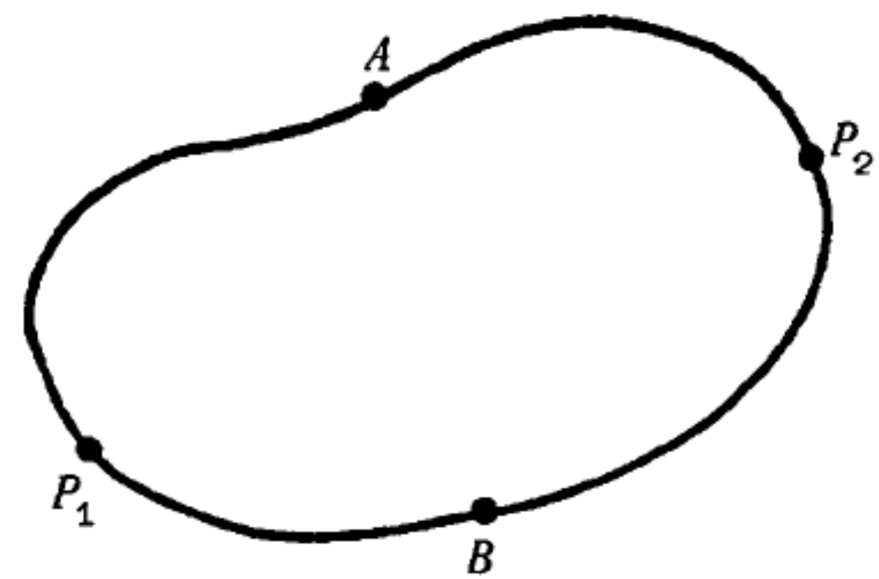
From part (*b*), $\phi(x,y,z) = x^2y + xz^3 + \text{constant}$.

Then work done $= \phi(3,1,4) - \phi(1,-2,1) = 202$.

Important massage

. Prove that if $\int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r}$ is independent of the path joining any two points P_1 and P_2 in a given region, then $\oint \mathbf{F}\cdot d\mathbf{r} = 0$ for all closed paths in the region and conversely.

Let $P_1AP_2BP_1$ (see adjacent figure) be a closed curve. Then



$$\oint \mathbf{F}\cdot d\mathbf{r} = \int_{P_1AP_2BP_1} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} + \int_{P_2BP_1} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} - \int_{P_1BP_2} \mathbf{F}\cdot d\mathbf{r} = 0$$

since the integral from P_1 to P_2 along a path through A is the same as that along a path through B, by hypothesis.

Conversely if $\oint \mathbf{F}\cdot d\mathbf{r} = 0$, then

$$\int_{P_1AP_2BP_1} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} + \int_{P_2BP_1} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} - \int_{P_1BP_2} \mathbf{F}\cdot d\mathbf{r} = 0$$

so that, $$\int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1BP_2} \mathbf{F}\cdot d\mathbf{r}.$$

(a) Show that a necessary and sufficient condition that $F_1\,dx + F_2\,dy + F_3\,dz$ be an exact differential is that $\nabla\times\mathbf{F} = \mathbf{0}$ where $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$.

(b) Show that $(y^2z^3\cos x - 4x^3z)\,dx + 2z^3y\sin x\,dy + (3y^2z^2\sin x - x^4)\,dz$ is an exact differential of a function ϕ and find ϕ.

(a) Suppose $F_1\,dx + F_2\,dy + F_3\,dz = d\phi = \dfrac{\partial\phi}{\partial x}dx + \dfrac{\partial\phi}{\partial y}dy + \dfrac{\partial\phi}{\partial z}dz$, an exact differential. Then since x, y and z are independent variables,

$$F_1 = \frac{\partial\phi}{\partial x}, \qquad F_2 = \frac{\partial\phi}{\partial y}, \qquad F_3 = \frac{\partial\phi}{\partial z}$$

and so $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = \nabla\phi$. Thus $\nabla\times\mathbf{F} = \nabla\times\nabla\phi = \mathbf{0}$.

Conversely if $\nabla\times\mathbf{F} = \mathbf{0}$ then by Problem 11, $\mathbf{F} = \nabla\phi$ and so $\mathbf{F}\cdot d\mathbf{r} = \nabla\phi\cdot d\mathbf{r} = d\phi$, i.e. $F_1\,dx + F_2\,dy + F_3\,dz = d\phi$, an exact differential.

(b) $\mathbf{F} = (y^2z^3\cos x - 4x^3z)\mathbf{i} + 2z^3y\sin x\,\mathbf{j} + (3y^2z^2\sin x - x^4)\mathbf{k}$ and $\nabla\times\mathbf{F}$ is computed to be zero, so that by part (a)

$$(y^2z^3\cos x - 4x^3z)\,dx + 2z^3y\sin x\,dy + (3y^2z^2\sin x - x^4)\,dz = d\phi$$

By any of the methods of Problem 12 we find $\phi = y^2z^3\sin x - x^4z + \text{constant}$.

Example

$\mathbf{F} = xy\,\mathbf{i} - z\,\mathbf{j} + x^2\,\mathbf{k}$ and C is the curve $x = t^2,\ y = 2t,\ z = t^3$ from $t = 0$ to $t = 1$ evaluate the line integral $\int_C \mathbf{F} \times d\mathbf{r}$.

Along C, $\quad \mathbf{F} = xy\,\mathbf{i} - z\,\mathbf{j} + x^2\,\mathbf{k} = 2t^3\,\mathbf{i} - t^3\,\mathbf{j} + t^4\,\mathbf{k}.$

Then $\quad \mathbf{F} \times d\mathbf{r} = (2t^3\,\mathbf{i} - t^3\,\mathbf{j} + t^4\,\mathbf{k}) \times (2t\,\mathbf{i} + 2\mathbf{j} + 3t^2\,\mathbf{k})\,dt$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} dt = [(-3t^5 - 2t^4)\mathbf{i} + (2t^5 - 6t^5)\mathbf{j} + (4t^3 + 2t^4)\mathbf{k}]\,dt$$

and

$$\int_C \mathbf{F} \times d\mathbf{r} = \mathbf{i}\int_0^1 (-3t^5 - 2t^4)\,dt + \mathbf{j}\int_0^1 (-4t^5)\,dt + \mathbf{k}\int_0^1 (4t^3 + 2t^4)\,dt$$

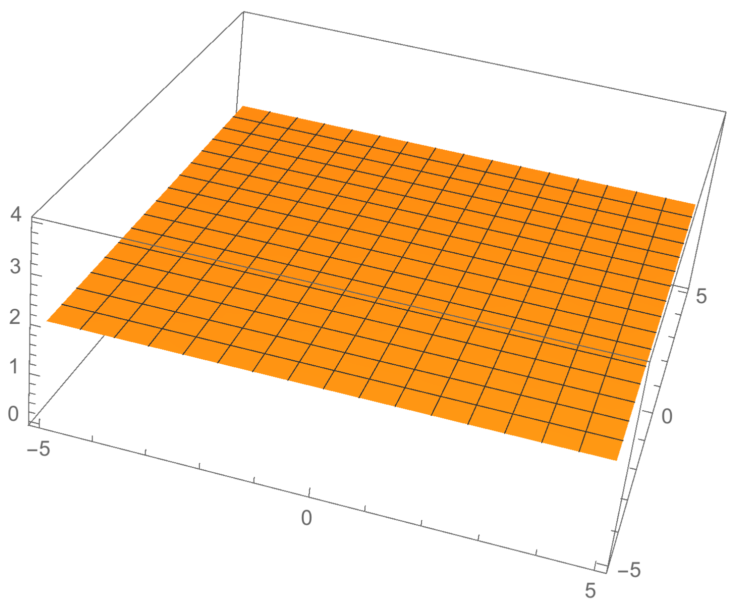
$$= -\frac{9}{10}\,\mathbf{i} - \frac{2}{3}\,\mathbf{j} + \frac{7}{5}\,\mathbf{k}$$

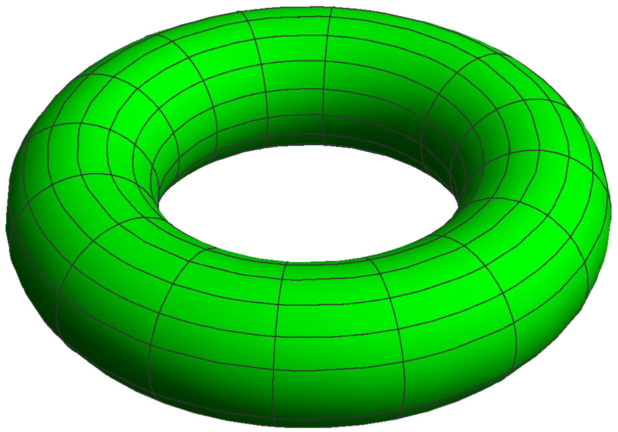
Surface Integration

Surface Integrals: How do we define a surface?

$z = f(x,y)$ or $x = f(y,z)$ or $y = f(x,z)$ is one of the standard form to represent surfaces.

Examples

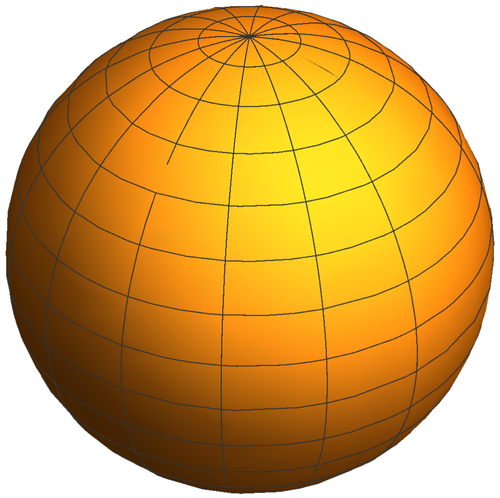




1. $z = \text{constant}$ is a plane parallel to xy plane.

2. $z^2 = a^2 - (c - \sqrt{x^2+y^2})^2$ is a torus.

Another way to represent: $f(x,y,z) = \text{constant}$



The sphere

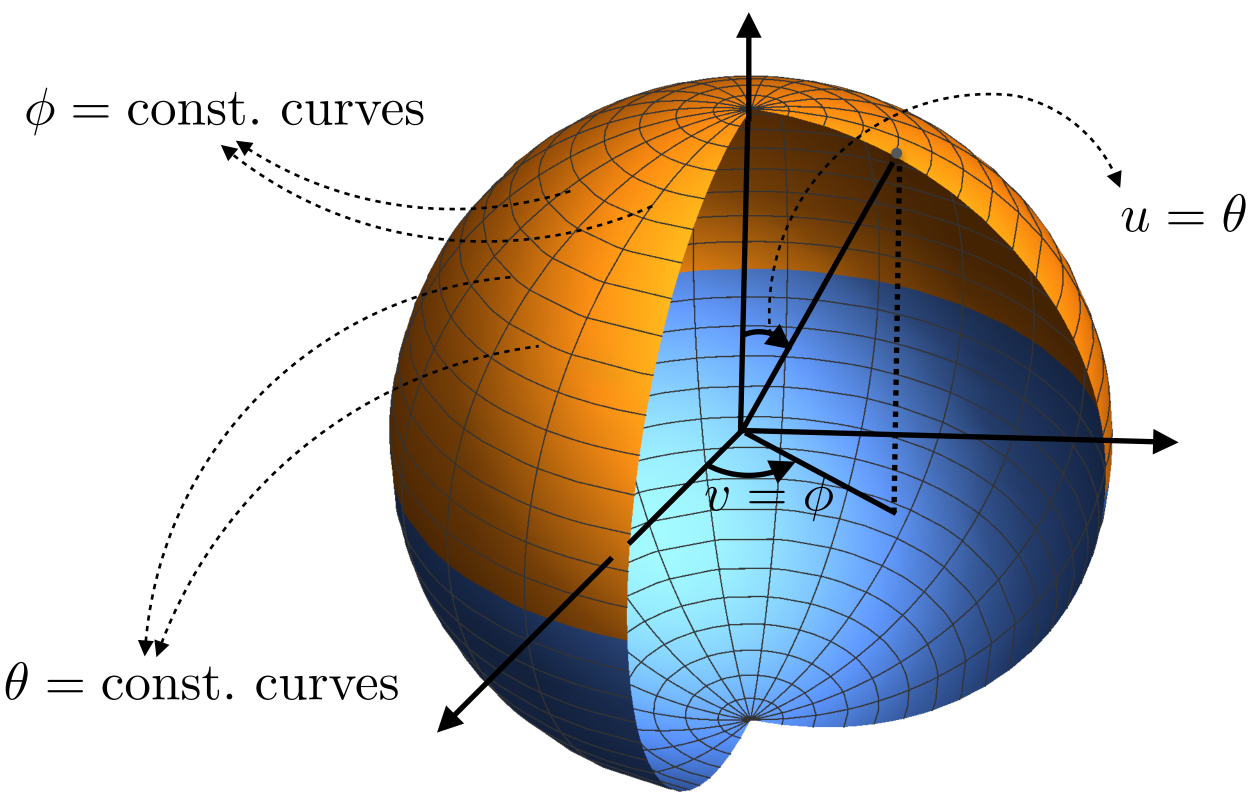
$x^2 + y^2 + z^2 = a^2$

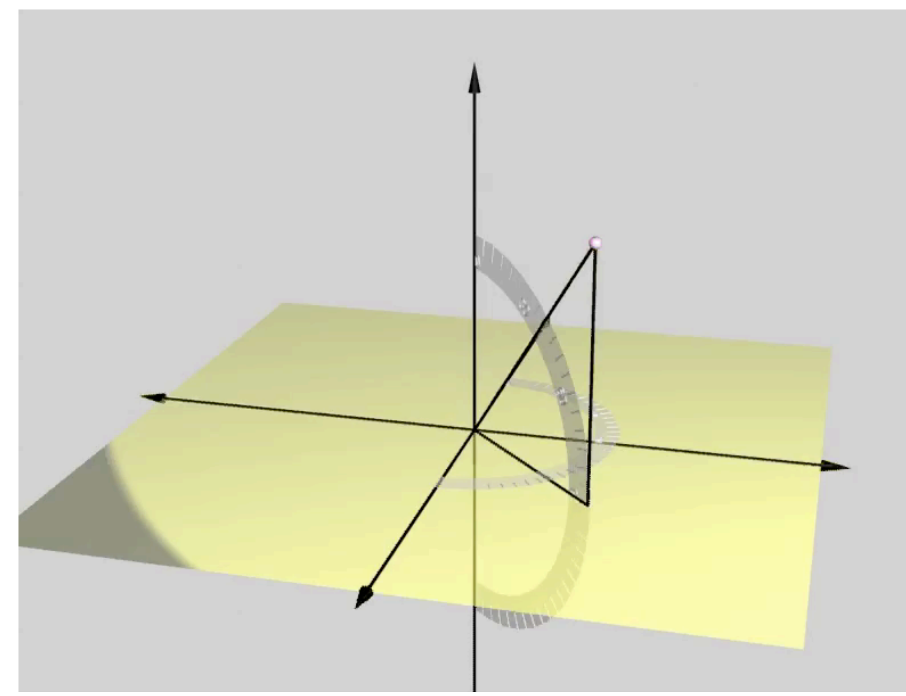
Parametric representation of a surface

Most generally, any arbitrary surface can be parametrically defined in terms of two real, orthogonal parameters (u, v) and real valued functions $x(u,v), y(u,v), z(u,v)$.

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

Example





We parametrise the sphere as

$$\vec{r}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta);\ \ 0 \le \theta \le \pi,\ \ 0 \le \phi \le 2\pi$$

$\star$ For $\theta = \frac{\pi}{2}$ (i.e. const.), $\vec{r}(\theta = \frac{\pi}{2}, \phi) = (\cos\phi, \sin\phi, 0) \implies$ Circle (lattitude)

$\star$ For $\phi = \frac{\pi}{2}$ (i.e. const.), $\vec{r}(\theta, \phi = \frac{\pi}{2}) = (0, \sin\theta, \cos\theta) \implies$ Circle (longitude)

$u =$ const. or $v =$ const. curves $\implies$ Parametric Curves

Concept of area as a vector

Imagine a tiny area (like a postage stamp) in 3 dimensions at some location $\vec{r}$. What can I do to specify it?

- how big it is?

da square meters (say).

- in which plane it lies?

in the xy plane (say)

it lies perpendicular to z axis

$\Longrightarrow$ A vector $d\vec{a}$, of magnitude da and direction along the z axis can be associated with this area.

But, there are two ways to draw $\perp$ to xy plane: up or down the z axis.

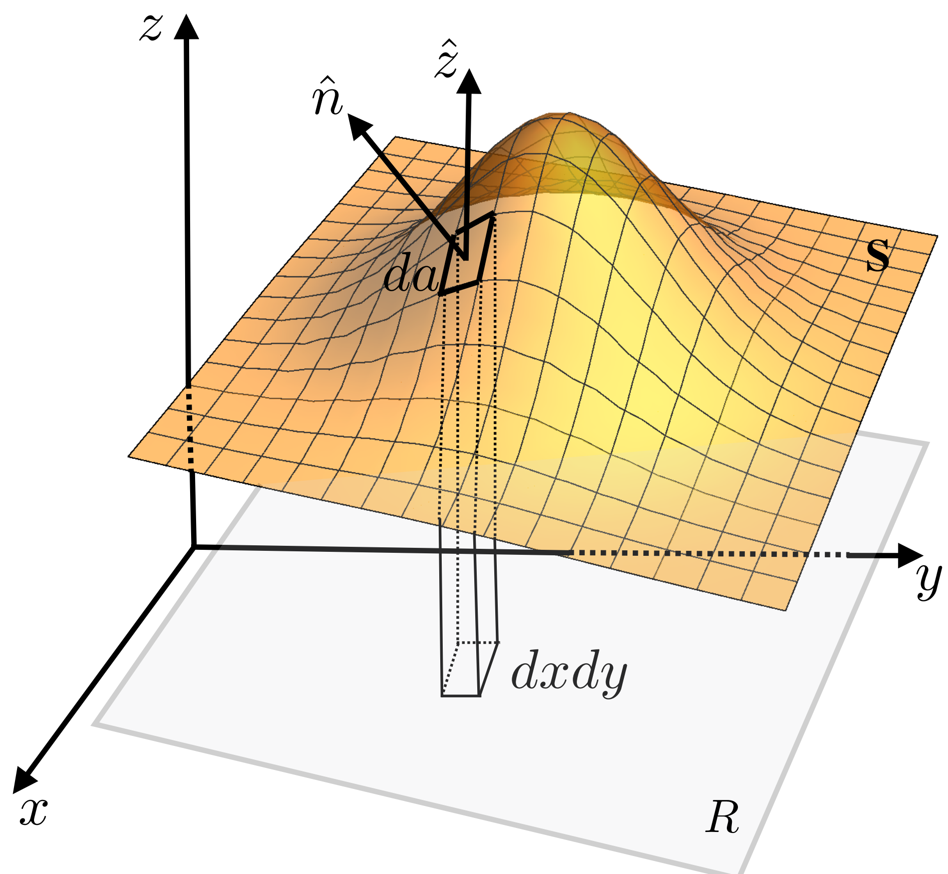
To further specify the area, to make it an oriented one, we draw arrows that run around the perimeter of the area in one of the two possible directions.

Area vector will point in the direction following the right hand thumb rule.

Only a planar area can be represented as a vector. Non-Planar areas like a hemisphere can not be represented by a single vector.

The use of right hand rule in defining areas might remind you of the cross product and indeed that is true as we will see soon.

Elementary area on a surface



- Let S be a smooth surface: $z = f(x, y)$.
- Project it on xy-plane: R be the projection.
- Choose an elementary area da on S and let $\hat{n}$ be a unit vector perpendicular to it.
- Projection of da on xy-plane is $dxdy$.

$$\therefore dxdy = |\hat{n}.\hat{z}|\ da$$

$$da = \frac{dxdy}{|\hat{n}.\hat{z}|}$$

- Hence we can denote da as vector area

$$d\vec{a} = \left(\frac{dxdy}{|\hat{n}.\hat{z}|}\right)\hat{n} = \hat{n}da$$

For an open two-sided surface, the "outward" normal shows the direction for the surface. Open surfaces are bounded by curves and "outward" normal is defined by the right hand rule-if the bounding curve is traversed in the direction of rotation of a right handed screw, the direction in which the head of screw moves is the direction of outward normal.

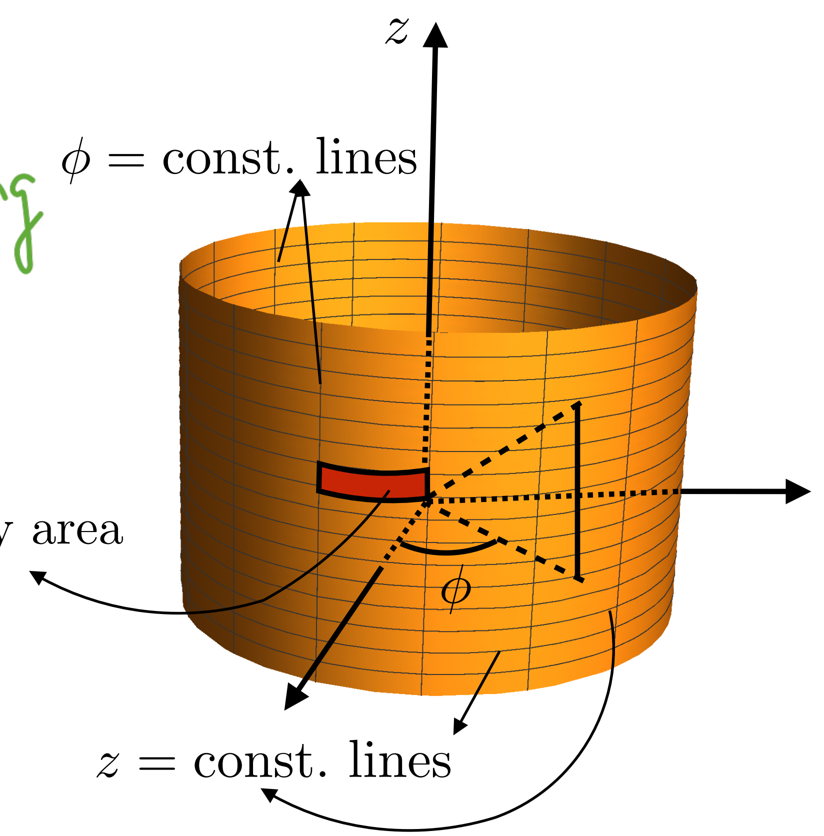
Elementary area: Parametrised surface

Suppose we have a cylinder of radius $R = 3$ units and parametrised by ϕ, z.

$$\vec{r}(\phi, z) = (3\cos\phi, 3\sin\phi, z);\ 0 \le \phi \le 2\pi, \quad -2 \le z \le 2.$$

Take a point on the cylinder at $\phi = 0, z = 1$ and then $\vec{r}(\phi = 0, z = 1) = (3, 0, 1)$

Choose an elementary area (shown in red)



Then $\overrightarrow{AB} = \underbrace{(\partial\vec{r}/\partial z)}_{\text{As } \phi \text{ is not changing}} dz$ and $\overrightarrow{AC} = \overbrace{(\partial\vec{r}/\partial\phi)}^{z \text{ is not changing}} d\phi$

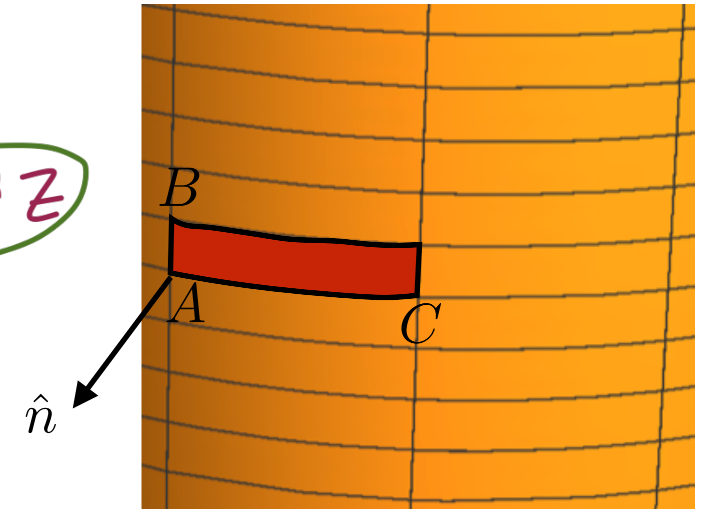
Normal vector at A: $\vec{n} = (\frac{\partial\vec{r}}{\partial\phi}) \times (\frac{\partial\vec{r}}{\partial z})$

Scalar area element $\quad da = |\overrightarrow{AC} \times \overrightarrow{AB}| = |\vec{n}| d\phi dz$

Elementary vector area $\quad d\vec{a} = \left(\frac{\vec{n}}{|\vec{n}|}\right) da = \vec{n} d\phi\, dz$

$= \hat{n}\, |\vec{n}|\, d\phi\, dz$

$= \hat{n}\, da$



H.W. : Complete the calculation for $\vec{S} = (3, 0, 1)$.

Home Work.

$$\vec{n} = \frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial z}$$

$$\vec{r} = 3\cos\varphi\,\hat{x} + 3\sin\varphi\,\hat{y} + z\,\hat{z}$$

$$\therefore \frac{\partial \vec{r}}{\partial \varphi} = -3\sin\varphi\,\hat{x} + 3\cos\varphi\,\hat{y}$$

$$\frac{\partial \vec{r}}{\partial z} = \hat{z}$$

$$\therefore \vec{n} = -3\sin\varphi\,(\hat{x}\times\hat{z}) + 3\cos\varphi\,(\hat{y}\times\hat{z})$$

$$= -3\sin\varphi\,(-\hat{y}) + 3\cos\varphi\,\hat{x}$$

$$(\varphi = 0,\ z = 1) \longrightarrow \vec{0} + 3\hat{x} = 3\hat{x}$$

$$\therefore \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\hat{x}}{3} = \hat{x}$$

$$\therefore d\vec{a} = \hat{n}\,da = \hat{x}\,d\varphi\,dz\,|\vec{n}|$$

$$= \hat{x}\,3\,d\phi\,dz$$

Surface integrals

For scalar fields

If we have a surface parametrised by $\vec{r}(u,v)$, then the surface integral of a scalar field is given by

$$\int_{\mathcal{S}} T\ da = \int_R T(\vec{r}(u,v))da = \int_R T(\vec{r}(u,v)) \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dudv$$

For vector fields

If we have a vector field $\vec{v}$, then the surface integral of the vector field over the surface parametrised by $\vec{r}(u,v)$ is

$$\int_{\mathcal{S}} \vec{v}.d\vec{a} = \int_R \vec{v}(\vec{r}(u,v)).\hat{n}da = \int_R \vec{v}(\vec{r}(u,v)). \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| \hat{n}dudv$$

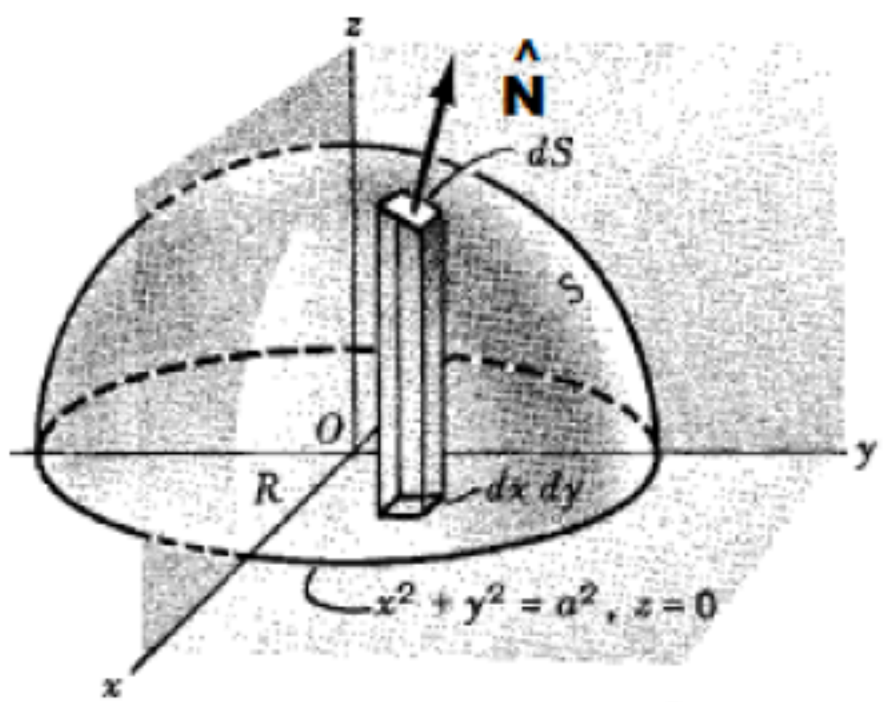
If $\overrightarrow{v}$ is the velocity of a fluid, then $\rho \overrightarrow{v} \cdot d\overrightarrow{a}$ is the mass passes through the area $d\overrightarrow{a}$ per unit time. Therefore the above is called as the flux of $\overrightarrow{v}$.

Example

The area of the upper hemispherical surface of radius a, i.e.,

$$F(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0;\ z > 0$$

.



A normal to $x^2 + y^2 + z^2 = a^2$ is

$$\nabla(x^2+y^2+z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$$

Then the unit normal is

$$\hat{\mathbf{N}} = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}$$

$$\text{Area} = \iint_R \frac{dx\,dy}{|\hat{\mathbf{N}}\cdot\mathbf{k}|} = \iint_R \frac{dx\,dy}{z/a} = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{dy\,dx}{\sqrt{a^2-x^2-y^2}}$$

$$= \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \frac{\rho\,d\rho\,d\phi}{\sqrt{a^2-\rho^2}} = 2\pi a^2$$

where $x = \rho\cos\phi$, $y = \rho\sin\phi$ and $dy\,dx$ is replaced by $\rho\,d\rho\,d\phi$.

Example:

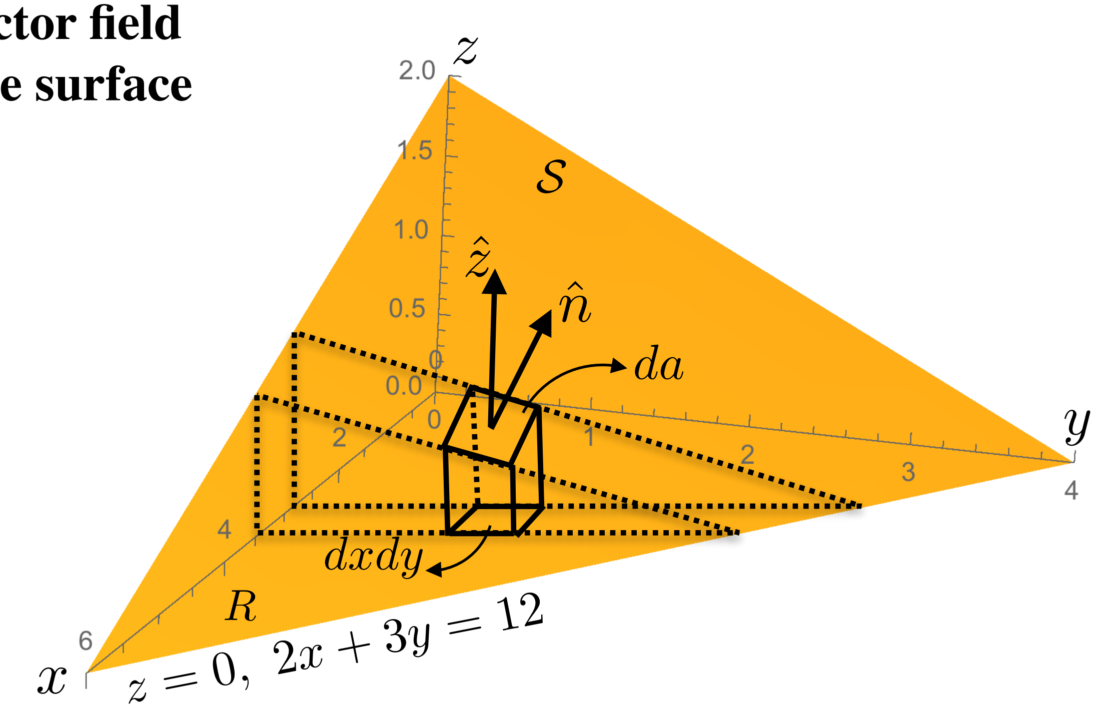
Evaluate $\int_{\mathcal{S}} \vec{A}.d\vec{a}$, where $\vec{A} = 18z\hat{x} - 12\hat{y} + 3y\hat{z}$ and $\mathcal{S}$ is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant.

The surface $\mathcal{S}$ and its projection R are shown in figure.

We have already seen that

Flux of vector field through the surface

$$\int_{\mathcal{S}} \vec{A}.d\vec{a} = \int_{\mathcal{S}} \vec{A}.\hat{n}da = \int_{R} \vec{A}.\hat{n}\frac{dxdy}{|\hat{n}.\hat{z}|}$$



To find the normal to the surface:

A vector perpendicular to $2x + 3y + 6z = 12$ is given by $\vec{\nabla}(2x + 3y + 6z) = 2\hat{x} + 3\hat{y} + 6\hat{z}$.

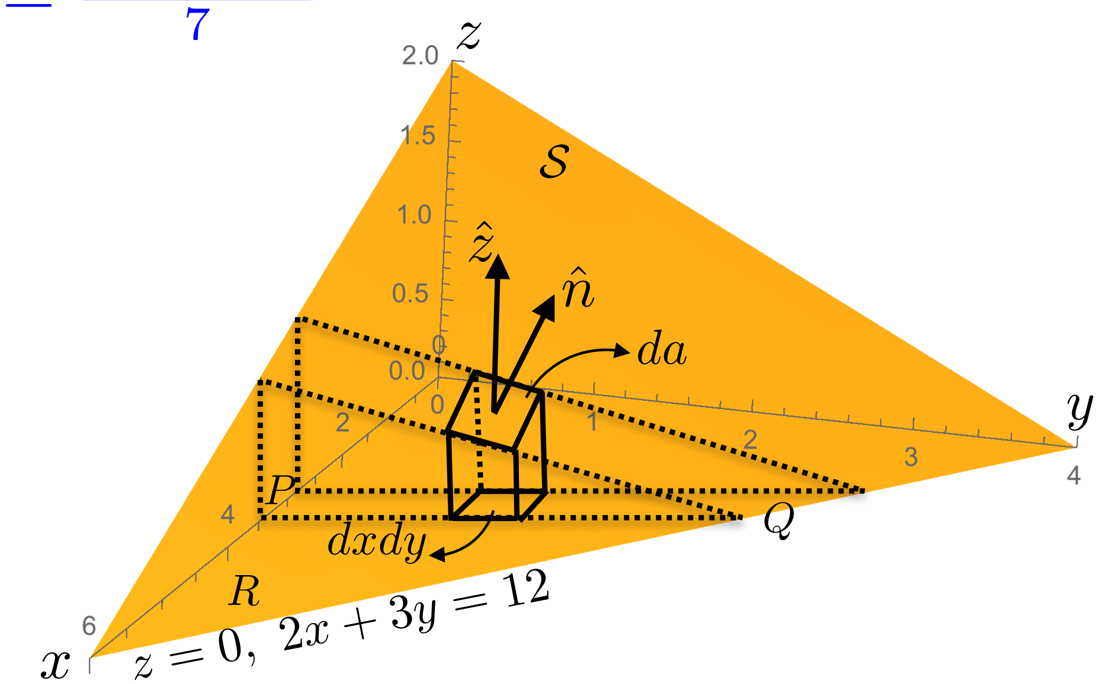
Then a unit normal to any point on $\mathcal{S}$ is: $\hat{n} = \dfrac{2\hat{x} + 3\hat{y} + 6\hat{z}}{\sqrt{2^2 + 3^2 + 6^2}} = \dfrac{2}{7}\hat{x} + \dfrac{3}{7}\hat{y} + \dfrac{6}{7}\hat{z}$

Example (contd.):

Then $\hat{n}.\hat{z} = (\frac{2}{7}\hat{x} + \frac{3}{7}\hat{y} + \frac{6}{7}\hat{z}).\hat{z} = \frac{6}{7}$ and $\frac{dxdy}{|\hat{n}.\hat{z}|} = \frac{7}{6}dxdy$.

Also, $\vec{A}.\hat{n} = (18z\hat{x} - 12\hat{y} + 3y\hat{z}).(\frac{2}{7}\hat{x} + \frac{3}{7}\hat{y} + \frac{6}{7}\hat{z}) = \frac{(36-12x)}{7}$

, using the fact that $z = \frac{12-2x-3y}{6}$.



$$\int_{\mathcal{S}} \vec{A}.\hat{n}da = \int_R \vec{A}.\hat{n}\frac{dxdy}{|\hat{n}.\hat{z}|}$$
$$= \int_R \left(\frac{36-12x}{7}\right)\frac{7}{6}dxdy$$

The integral is a double integral and to do it, first keep x fixed and integrate with respect to y from $y = 0$ (P in the figure) to $y = \frac{12-2x}{3}$ (Q in the figure); then integrate w.r.to x from $x = 0$ to $x = 6$.

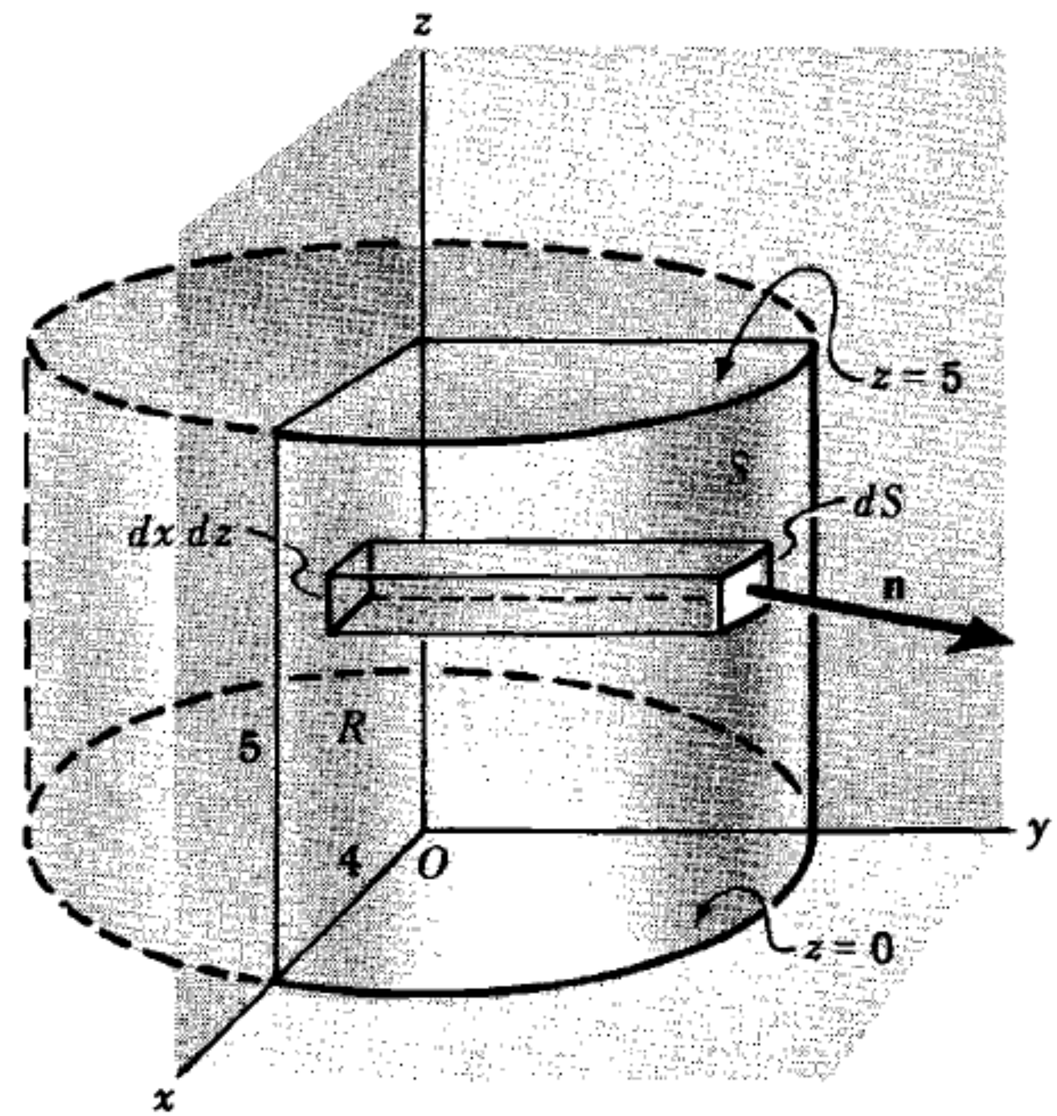
$$\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3}(6-2x)dxdy = \int_{x=0}^{6}(24 - 12x + \frac{4x^2}{3})dx = 24$$

Example

Evaluate $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$, where $\mathbf{A} = z\,\mathbf{i} + x\,\mathbf{j} - 3y^2 z\,\mathbf{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Project S on the xz plane as in the figure below and call the projection R. Note that the projection of S on the xy plane cannot be used here. Then

$$\iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_R \mathbf{A}\cdot\mathbf{n}\,\frac{dx\,dz}{|\mathbf{n}\cdot\mathbf{j}|}$$



A normal to $x^2+y^2=16$ is $\nabla(x^2+y^2) = 2x\,\mathbf{i}+2y\,\mathbf{j}$. Thus the unit normal to S as shown in the adjoining figure, is

$$\mathbf{n} = \frac{2x\,\mathbf{i}+2y\,\mathbf{j}}{\sqrt{(2x)^2+(2y)^2}} = \frac{x\,\mathbf{i}+y\,\mathbf{j}}{4}$$

since $x^2+y^2=16$ on S.

$$\mathbf{A}\cdot\mathbf{n} = (z\,\mathbf{i}+x\,\mathbf{j}-3y^2z\,\mathbf{k})\cdot\left(\frac{x\,\mathbf{i}+y\,\mathbf{j}}{4}\right) = \frac{1}{4}(xz+xy)$$

$$\mathbf{n}\cdot\mathbf{j} = \frac{x\,\mathbf{i}+y\,\mathbf{j}}{4}\cdot\mathbf{j} = \frac{y}{4}.$$

Then the surface integral equals

$$\iint_R \frac{xz+xy}{y}\,dx\,dz = \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{16-x^2}}+x\right)dx\,dz = \int_{z=0}^{5}(4z+8)\,dz = 90$$

Example

Evaluate $\iint_S \phi \mathbf{n}\, dS$ where $\phi = \frac{3}{8}xyz$ and S is the surface of Problem 20.
earlier example.

We have
$$\iint_S \phi \mathbf{n}\, dS = \iint_R \phi \mathbf{n} \frac{dx\, dz}{|\mathbf{n}\cdot\mathbf{j}|}$$

Using $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{4}$, $\mathbf{n}\cdot\mathbf{j} = \dfrac{y}{4}$ as in Problem 20, this last integral becomes
earlier example

$$\iint_R \frac{3}{8}xz\,(x\,\mathbf{i} + y\,\mathbf{j})\; dx\, dz = \frac{3}{8}\int_{z=0}^{5}\int_{x=0}^{4} (x^2 z\,\mathbf{i} + xz\sqrt{16-x^2}\,\mathbf{j})\; dx\, dz$$

$$= \frac{3}{8}\int_{z=0}^{5} (\frac{64}{3}z\,\mathbf{i} + \frac{64}{3}z\,\mathbf{j})\, dz = 100\,\mathbf{i} + 100\,\mathbf{j}$$

Volume Integration

Volume Integrals

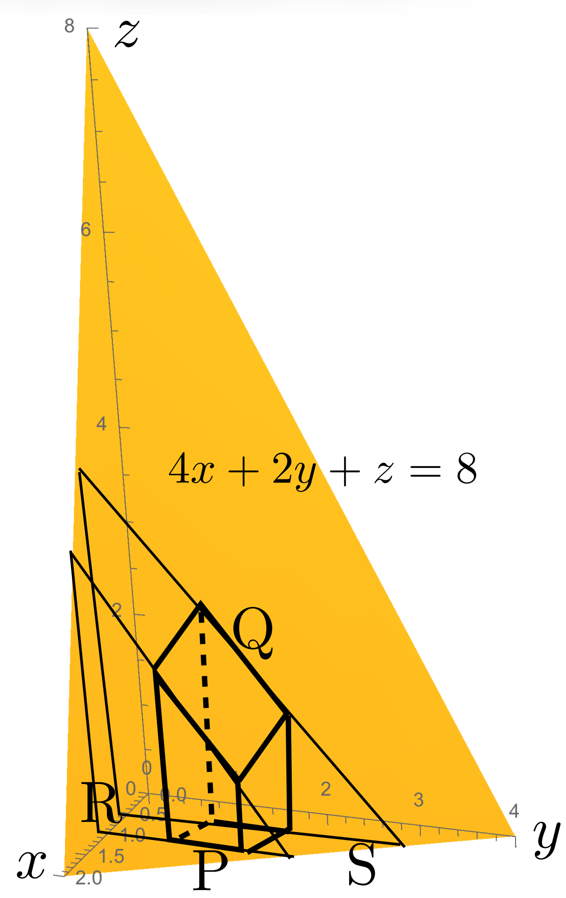
Consider a closed surface in space enclosing a volume. Then,

$$\int_{\mathcal{V}} \vec{v} d\tau \text{ and } \int_{\mathcal{V}} \mathrm{T} d\tau$$

are examples of volume integrals. Here $d\tau$ represents elementary volume.

Example:

Let $\phi = 45x^2y$ and let $\mathcal{V}$ denotes a closed region bounded by the planes $4x + 2y + z = 8$, $x = 0$, $y = 0$, $z = 0$. Evaluate the integral $\int_{\mathcal{V}} \phi d\tau$.



Strategy

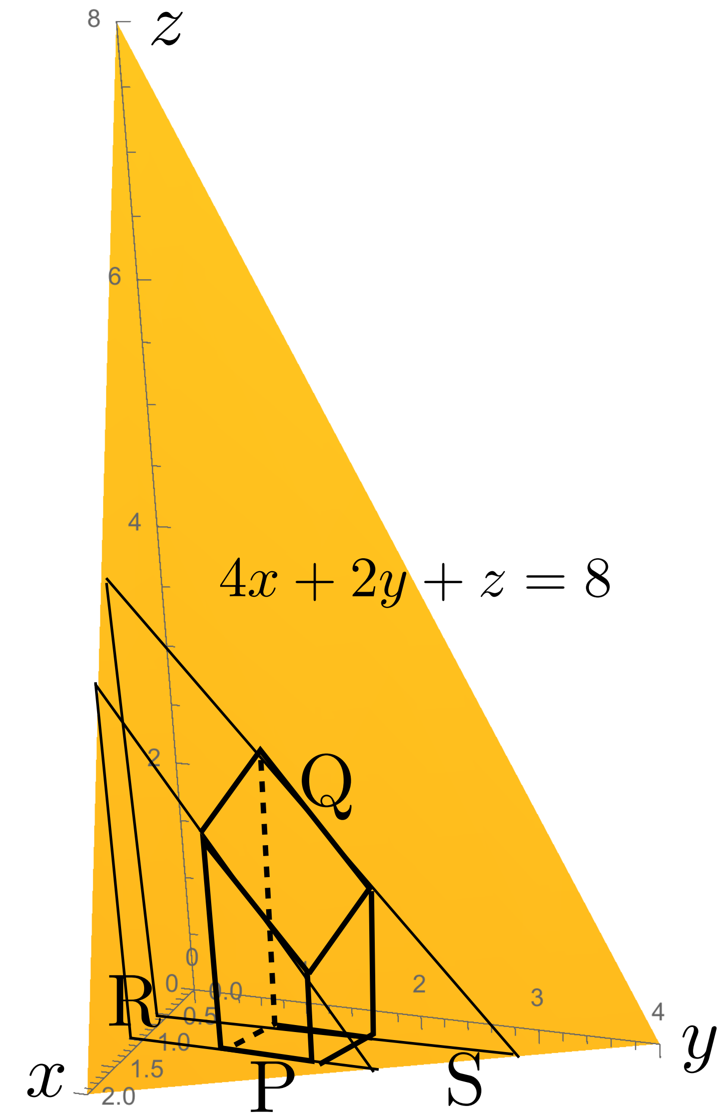
Keep x and y constant and integrate from $z = 0$ (base of the column PQ in figure) to $z = 8 - 4x - 2y$ (top of the column PQ)

Next keep x constant and integrate w.r.t y. This amounts to addition of columns having bases in the xy plane ($z = 0$) located anywhere from R (where $y = 0$) to S (where $4x + 2y = 8$ or $y = 4 - 2x$), and the integrations from $y = 0$ to $y = 4 - 2x$.

Volume Integrals

Finally add all slabs parallel to yz plane, which amounts to integration from $x = 0$ to $x = 2$.

$$
\begin{aligned}
\int_{\mathcal{V}} \phi d\tau &= \int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} 45x^2 y dz dy dx \\
&= 45 \int_{x=0}^{2} \int_{y=0}^{4-2x} x^2 y (8 - 4x - 2y) dy dx \\
&= 45 \int_{x=0}^{2} \frac{8}{3} x^2 (2-x)^3 dx \\
&= 120 \int_{x=0}^{2} x^2 (2-x)^3 dx \\
&= 120 \times \frac{16}{15} = 128
\end{aligned}
$$

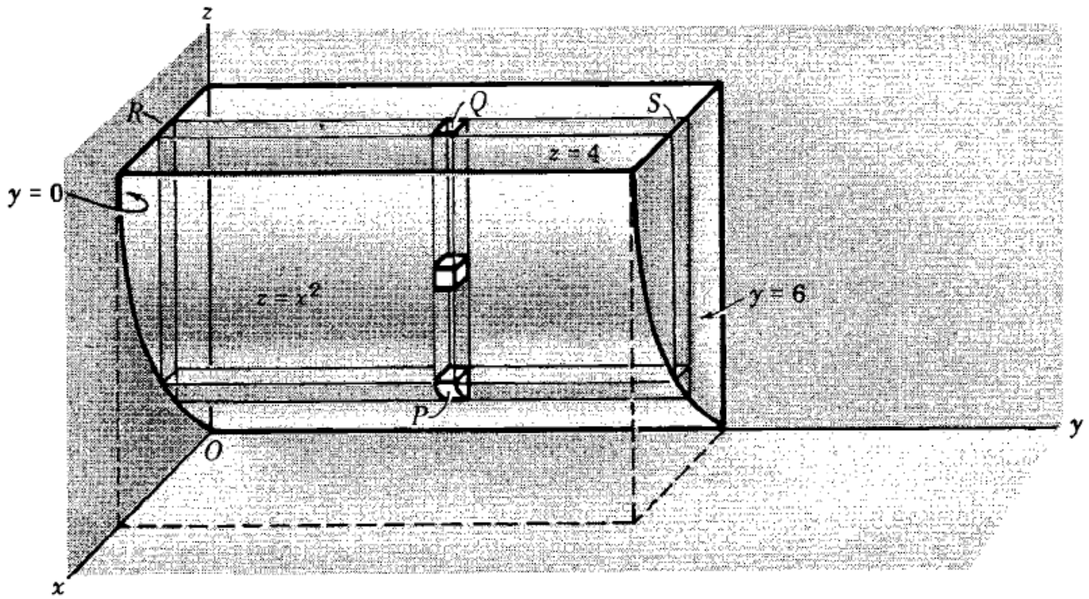


Physically the result can be interpreted as the mass of the region $\mathcal{V}$ in which the density varies according to the formula $\phi = 45x^2 y$.

Example

Let $\mathbf{F} = 2xz\,\mathbf{i} - x\,\mathbf{j} + y^2\,\mathbf{k}$. Evaluate $\displaystyle\iiint_V \mathbf{F}\,dV$ where V is the region bounded by the surfaces $x=0$, $y=0$, $y=6$, $z=x^2$, $z=4$.

The region V is covered (a) by keeping x and y fixed and integrating from $z=x^2$ to $z=4$ (base to top of column PQ), (b) then by keeping x fixed and integrating from $y=0$ to $y=6$ (R to S in the slab), (c) finally integrating from $x=0$ to $x=2$ (where $z=x^2$ meets $z=4$). Then the required integral is



$$\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} (2xz\,\mathbf{i} - x\,\mathbf{j} + y^2\,\mathbf{k})\,dz\,dy\,dx$$

$$= \mathbf{i}\int_0^2\int_0^6\int_{x^2}^4 2xz\,dz\,dy\,dx \;-\; \mathbf{j}\int_0^2\int_0^6\int_{x^2}^4 x\,dz\,dy\,dx \;+\; \mathbf{k}\int_0^2\int_0^6\int_{x^2}^4 y^2\,dz\,dy\,dx$$

$$= 128\,\mathbf{i} \;-\; 24\,\mathbf{j} \;+\; 384\,\mathbf{k}$$

What did we learn today:

• Generalisation of idea of integration in one variable to many variables.
• Parametrisation of curves and surfaces.
• Line integral of a scalar field f over a curve C whose parametric representation is given by the path $\vec{r}(t)$ is given by $\int_C f dr = \int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|dt$.
• Line integral of a vector field is given by $\int_C \vec{F}.d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)).\vec{r}\,'(t)dt$.
• Surface integral of a scalar field is given by $\int_{\mathcal{S}} f \;\; da = \int_R f(\vec{r}(u,v))da = \int_R f(\vec{r}(u,v)) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dudv$
• Surface integral of a scalar field is given by $\int_{\mathcal{S}} \vec{v}.d\vec{a} = \int_R \vec{v}(\vec{r}(u,v)).\hat{n}da = \int_R \vec{v}(\vec{r}(u,v)). \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \hat{n}dudv$.
• The volume integral of a scalar and a vector field can be written as $\int_{\mathcal{V}} T d\tau$ and $\int_{\mathcal{V}} \vec{v} d\tau$ respectively.