Physics II: Electromagnetism

PH 102

Lecture 3

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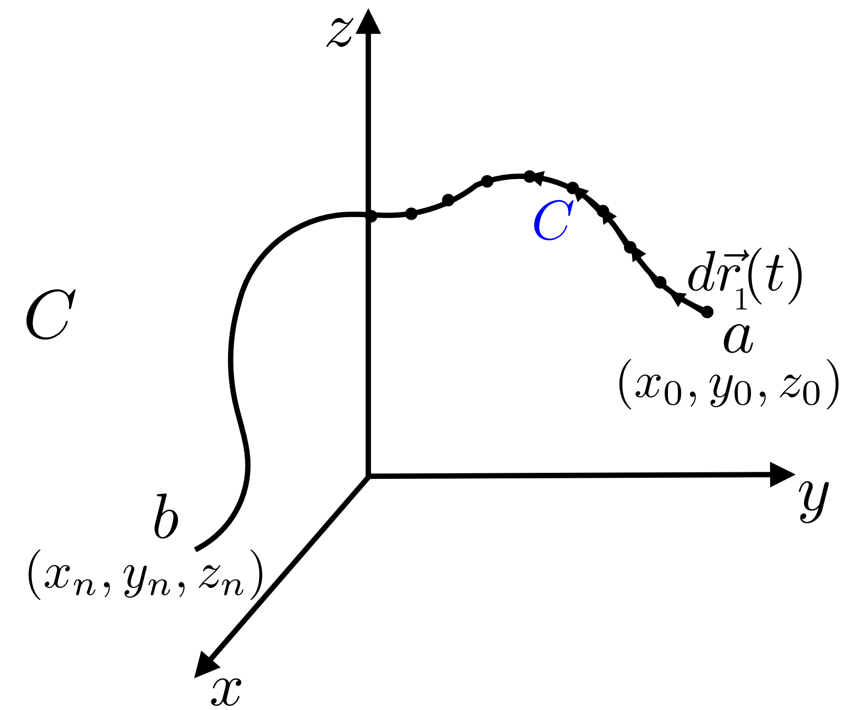
Fundamental theorem for gradients



- Scalar field $\phi(x, y, z)$.

- Start at a and move infinitesimal distance $d\vec{r}_1$ along path C to reach $\vec{r}_a + d\vec{r}_1$, $\vec{r}_a$ is position vector at a

Recall from first lecture that the scalar function ϕ will change by $\quad d\phi = (\vec{\nabla}\phi).d\vec{r}_1$

- Now, move a little further, by small displacement $d\vec{r}_2$.

- The change in ϕ will be $\vec{\nabla}\phi.d\vec{r}_2$.

- In this manner, proceeding by infinitesimal steps we reach point b. At each step computing gradient of ϕ (at that point) and dot it into the displacement.

- This gives the total change in ϕ as

Recall $\int_a^b \frac{df(x)}{dx} dx = f(b) - f(a)$

$$\int_a^b (\vec{\nabla}\phi).d\vec{r} = \phi(b) - \phi(a)$$

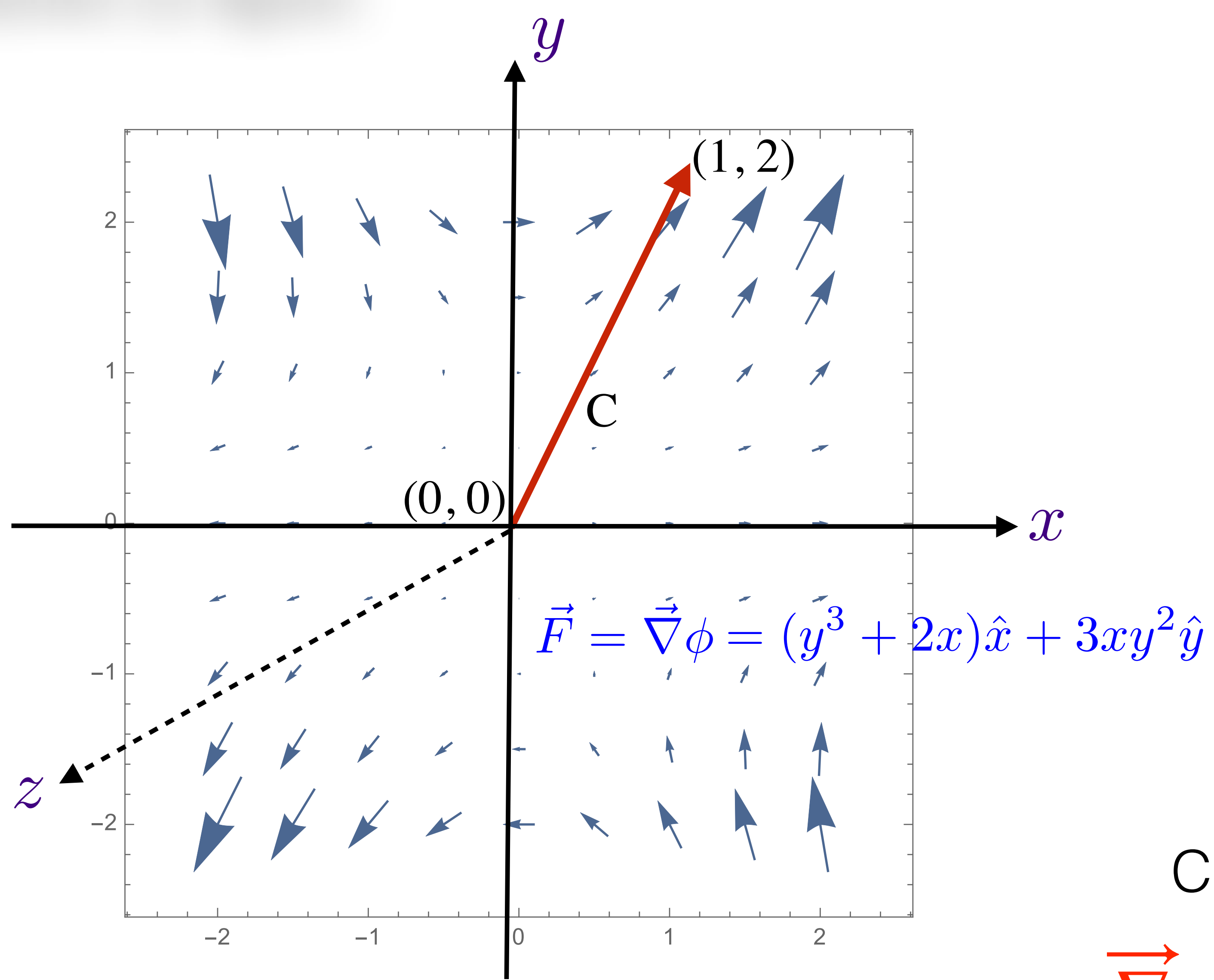
Fundamental theorem for gradients

Like the ordinary fundamental theorem of calculus, it says that the integral (here line integral) of a derivative (here the gradient) is given by the value of the function at the boundaries (a and b)

Example

Note that the $\vec{\nabla}\phi$ is a vector field and sometimes called the gradient field. The function ϕ will be called a potential function for the field.

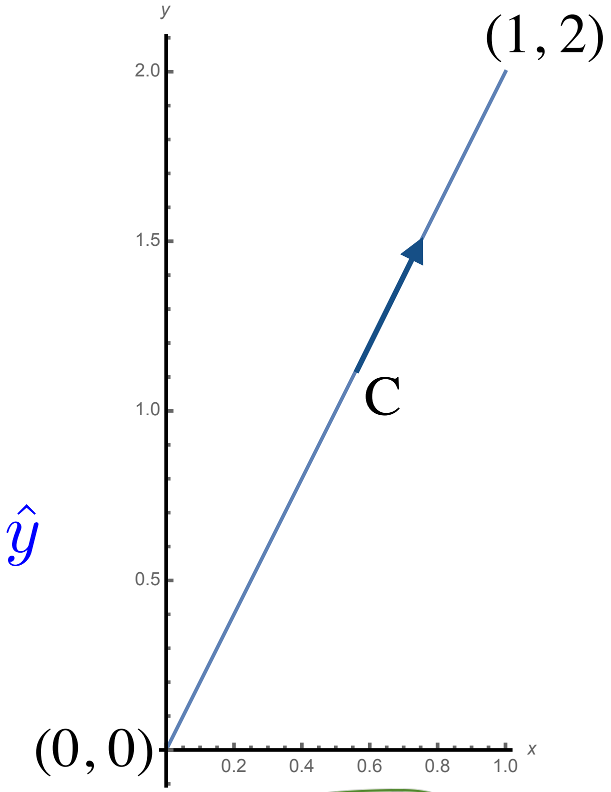
If $\vec{F} = \vec{\nabla}\phi$ is a gradient field where $\phi(x, y) = xy^3 + x^2$, then compute $I = \int_C \vec{F}.d\vec{r}$ along the curve C shown in figure.



Check:

$$\vec{\nabla} \times \vec{F} = \vec{0}$$

Example (contd.)



$$\phi(x, y) = xy^3 + x^2$$

- Method 1: Direct integration: (Line integral)

$$\vec{F} = \vec{\nabla}\phi = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(xy^3 + x^2) = (y^3 + 2x)\hat{x} + 3xy^2\hat{y}$$

Looking at the curve, we see that its equation is $y = 2x$, therefore, we parametrise as $x = t, y = 2t;\ 0 \leq t \leq 1$.

$\frac{dy}{dx} = \frac{2}{1}$

$$I = \int_C \vec{F}.d\vec{r} = \int_C (y^3 + 2x)dx + 3xy^2 dy = \int_0^1 (8t^3 + 2t)dt + 12t^3 2dt = \int_0^1 (32t^3 + 2t)dt = 9$$

- Method 2: Applying Gradient theorem

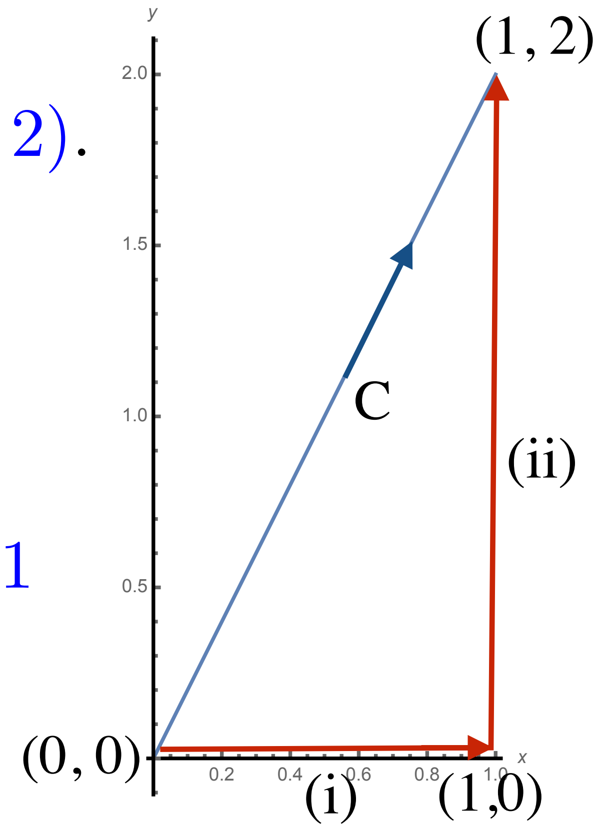
$$\int_C \vec{F}.d\vec{r} = \int_C \vec{\nabla}\phi.d\vec{r} = \phi(1, 2) - \phi(0, 0) = 9.$$

Example (contd.)

$$\vec{F} = \vec{\nabla}\phi = (y^3 + 2x)\hat{x} + 3xy^2\hat{y}$$

Same problem with a different choice of path

I could have taken a different path (in red): $(0,0) \to (1,0) \to (1,2)$.



- Path (i) $(0,0) \to (1,0)$: $y = 0$, $x : 0 \to 1$

$$I = \int_{(i)} \vec{F}.d\vec{r} = \int_{(i)} \vec{\nabla}\phi.d\vec{r} = \int_0^1 2x dx = 2.\frac{x^2}{2}\Big|_0^1 = 1$$

- Path (ii) $(1,0) \to (1,2)$: $x = 1, y : 0 \to 2$, i.e. $dx = 0$.

$$I = \int_{(ii)} \vec{F}.d\vec{r} = \int_{(ii)} \vec{\nabla}\phi.d\vec{r} = \int_0^2 3xy^2 dy = 3.1.\frac{y^3}{3}\Big|_0^2 = 8$$

- $\therefore$ to reach $(1,2)$ from $(0,0)$ via paths (i) & (ii): $(\int_{(i)} + \int_{(ii)})\vec{F}.d\vec{r} = 1 + 8 = 9$

Same as the one along curve C!

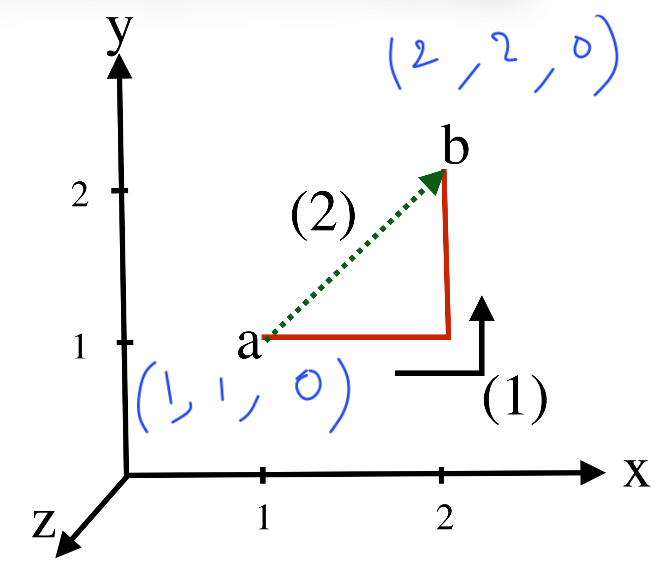
$\int_a^b \vec{\nabla}\phi.d\vec{r}$ **is independent of the path taken**

Some important corollaries

⋆ Line integrals in general depend on the path taken from a to b.

Example:

Calculate the line integral of $\vec{A} = y^2\hat{x} + 2x(y+1)\hat{y}$ from $(1,1,0)$ to $(2,2,0)$ along paths (1) and (2) in figure.



We need to calculate $\int \vec{A}.d\vec{r}$, where $d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$.

Red path (path 1):

Horizontal segment: $dy = dz = 0$, so $d\vec{r} = dx\hat{x}$, $y = 1$,

$$\vec{A}.d\vec{r} = y^2 dx = dx, \text{ Hence } \int_1^2 \vec{A}.d\vec{r} = \int_1^2 dx = 1.$$

Vertical segment: $dx = dz = 0$, so $d\vec{r} = dy\hat{y}$, $x = 2$,

$$\vec{A}.d\vec{r} = 2x(y+1)dy = 4(y+1)dy, \text{ so } \int \vec{A}.d\vec{r} = 4\int_1^2 (y+1)dy = 10$$

$$\text{Hence, by path (1), } \int_{(1,1,0)}^{(2,2,0)} \vec{A}.d\vec{r} = 1 + 10 = 11$$

Green path (path 2):

Here, $x = y$, i.e. $dx = dy$ and $dz = 0$. $d\vec{r} = dx\hat{x} + dy\hat{y}$

$$\therefore \vec{A}.d\vec{r} = x^2 dx + 2x(x+1)dx = (3x^2 + 2x)dx,$$

$$\int_{1,1,0}^{(2,2,0)} \vec{A}.d\vec{r} = \int_1^2 (3x^2 + 2x)dx = 10$$

Check:

$\vec{\nabla} \times \vec{A} \neq \vec{0}$

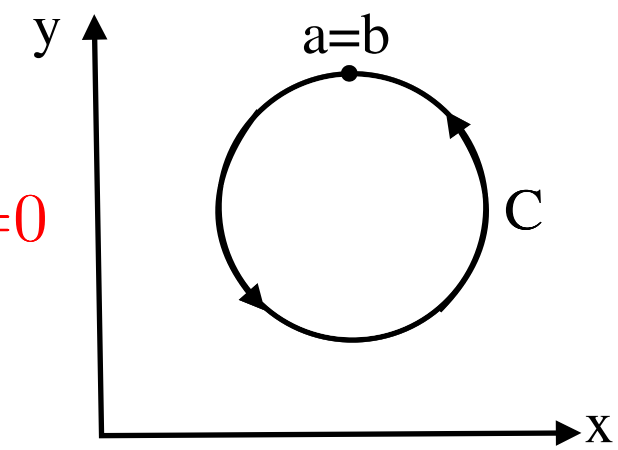
Some important corollaries

⋆ Gradients have the special property that their line integrals are path independent: $\int_a^b \vec{\nabla}\phi.d\vec{r}$ is independent of the path from a to b.

⋆ $\oint \vec{\nabla}\phi.d\vec{r} = 0$, since the beginning and end points are identical and hence $\phi(b) - \phi(a) = 0$.

Assume path independence and consider the closed path C in Figure 1. Since the starting point a and end point b are same, we get $\oint \vec{\nabla}\phi.d\vec{r} = \phi(b) - \phi(a) = 0$.

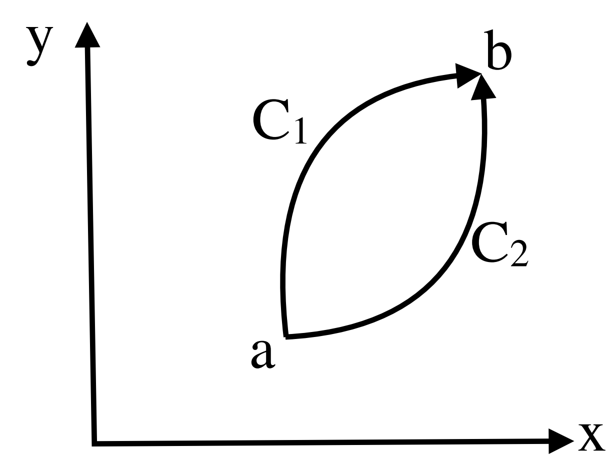
path independence $\Longrightarrow$ line integral around closed path=0



Assume $\oint \vec{\nabla}\phi.d\vec{r} = 0$ for any closed curve. If C_1 and C_2 are both paths between a and b, then $C_1 - C_2$ is a closed path. So by hypothesis

$$\oint_{C_1 - C_2} \vec{\nabla}\phi.d\vec{r} = \int_{C_1} \vec{\nabla}\phi.d\vec{r} - \int_{C_2} \vec{\nabla}\phi.d\vec{r} = 0$$

$$\Longrightarrow \int_{C_1} \vec{\nabla}\phi.d\vec{r} = \int_{C_2} \vec{\nabla}\phi.d\vec{r}$$



line integral around closed path=0 $\Longrightarrow$ path independence

Fundamental theorem for Divergence (Gauss's Theorem)

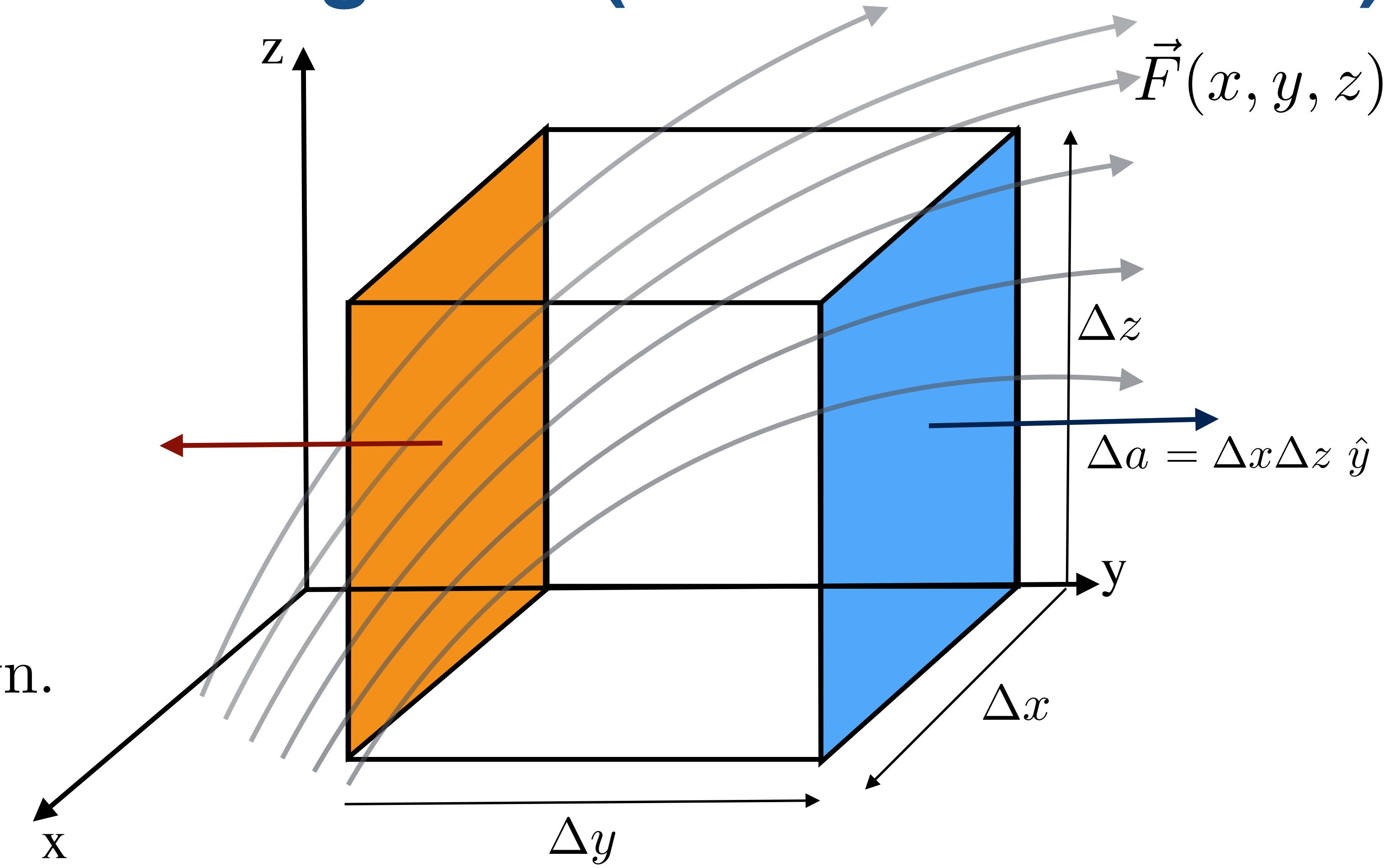
Consider a vector field $\vec{F}(x, y, z)$.

Say, flow of water...

We want to find out the flux of $\vec{F}(x, y, z)$.

To do that, take an infinitesimal volume element like a cube as shown.

..and look at the opposite faces (coloured in figure) first.



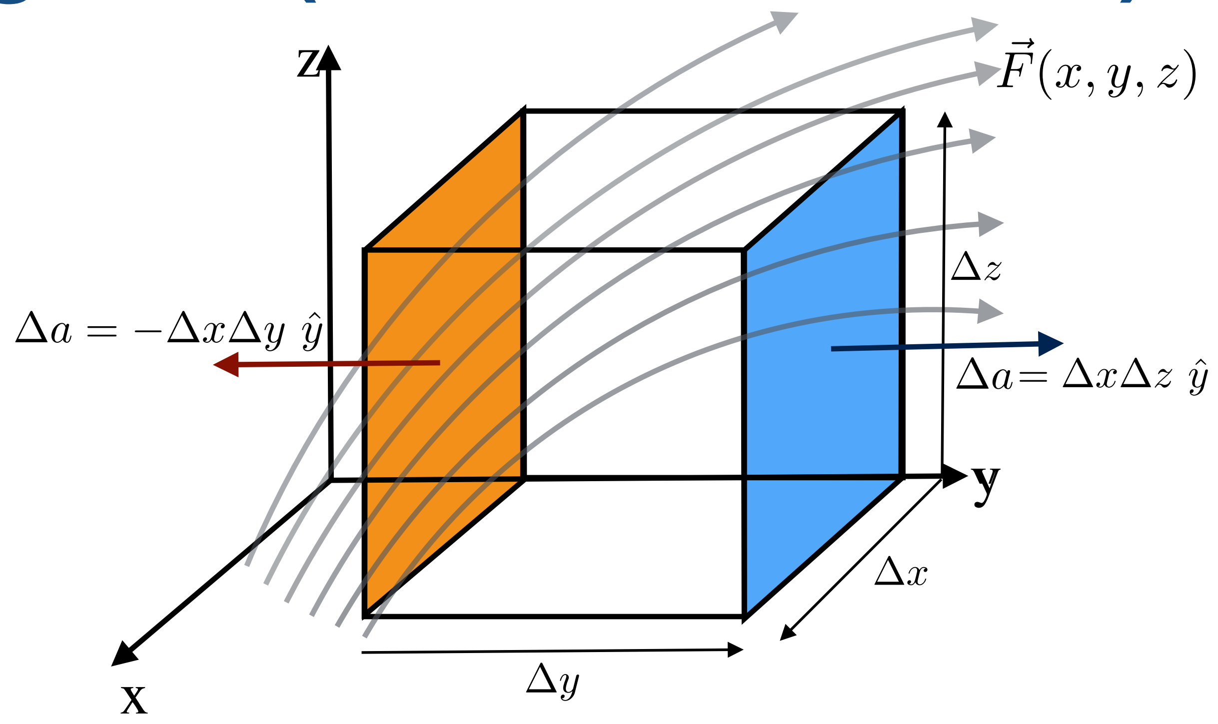
The left most face (orange) is at a fixed value of y and the incoming flux into this surface is $\vec{F}(x, y, z).\Delta\vec{a} = \vec{F}(x, y, z).\hat{n}\Delta a = (F_x\hat{x} + F_y\hat{y} + F_z\hat{z}).(-\Delta x\Delta z\hat{y}) = -F_y(x, y, z)\Delta x\Delta z$.

The right most face (blue) is at a fixed value of $y + \Delta y$ and the flux going out of this surface is $\vec{F}(x, y + \Delta y, z).\Delta\vec{a} = \vec{F}(x, y + \Delta y, z).\hat{n}\Delta a = (F_x\hat{x} + F_y\hat{y} + F_z\hat{z}).(\Delta x\Delta z\hat{y}) = F_y(x, y + \Delta y, z)\Delta x\Delta z$.

Fundamental theorem for Divergence (Gauss's Theorem)

So, the outward flux from the blue face is

$$\vec{F}(x, y+\Delta y, z).\Delta\vec{a} = F_y(x, y+\Delta y, z)\Delta x\Delta z$$
$$= \left(F_y + \frac{\partial F_y}{\partial y}\Delta y\right)\Delta x\Delta z$$



Net flux out of the box through these two opposite (orange and blue) faces

$$-F_y\Delta x\Delta z + \left(F_y + \frac{\partial F_y}{\partial y}\Delta y\right)\Delta x\Delta z = \frac{\partial F_y}{\partial y}\Delta y\Delta x\Delta z = \frac{\partial F_y}{\partial y}\Delta\tau$$

Net flux going out through the sides parallel to yz plane: $(\frac{\partial F_x}{\partial x})\Delta x\Delta y\Delta z$

Net flux going out through the sides parallel to xy plane: $(\frac{\partial F_z}{\partial z})\Delta z\Delta y\Delta x$

Summing over all the faces: $\vec{F}.\hat{n}\Delta a\Big|_{\text{all surfaces}} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\Delta x\Delta y\Delta z = \vec{\nabla}.\vec{F}\Delta\tau$

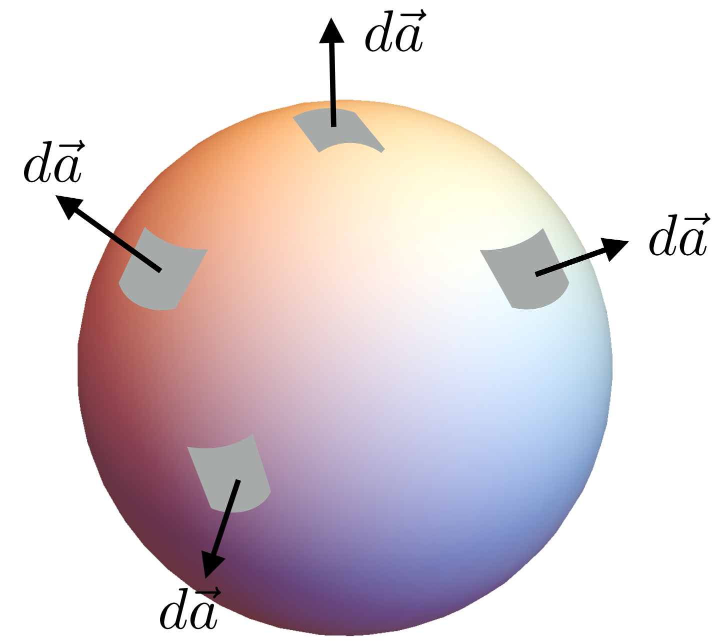
Flux over a closed surface can be written as a sum over the surfaces of elemental volumes that make the volume: $\sum\vec{F}.\hat{n}\Delta a = \sum(\vec{\nabla}.\vec{F})\Delta\tau$

In the limit $\Delta x, \Delta y, \Delta z \to 0$ $\int_{\mathcal{V}}(\vec{\nabla}.\vec{F})d\tau = \oint_{\mathcal{S}}\vec{F}.\hat{n}da$ Gauss's Divergence Theorem

Fundamental theorem for Divergence (Gauss's Theorem)

What does it mean?

We want to find the total outward flux of the vector field $\vec{F}(\vec{r})$ across a surface $\mathcal{S}$ that bounds a volume $\mathcal{V}$: $\oint_{\mathcal{S}} \vec{F}.d\vec{a}$



$d\vec{a}$ is

- normal to the local surface element
- points everywhere out of the volume

Gauss's theorem tells us that we can calculate the flux of vector field across a surface $\mathcal{S}$, by considering the total flux generated inside the volume $\mathcal{V}$.

Gauss's Theorem:

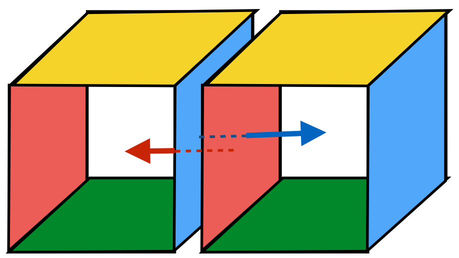
$$\int_{\mathcal{V}} (\vec{\nabla}.\vec{F}) d\tau = \oint_{\mathcal{S}} \vec{F}.\hat{n} da$$

Volume integrals are easier than the surface integrals: computational efficiency!

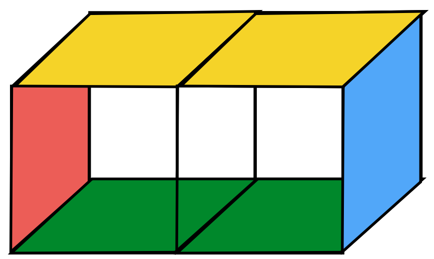
How to “see” this?

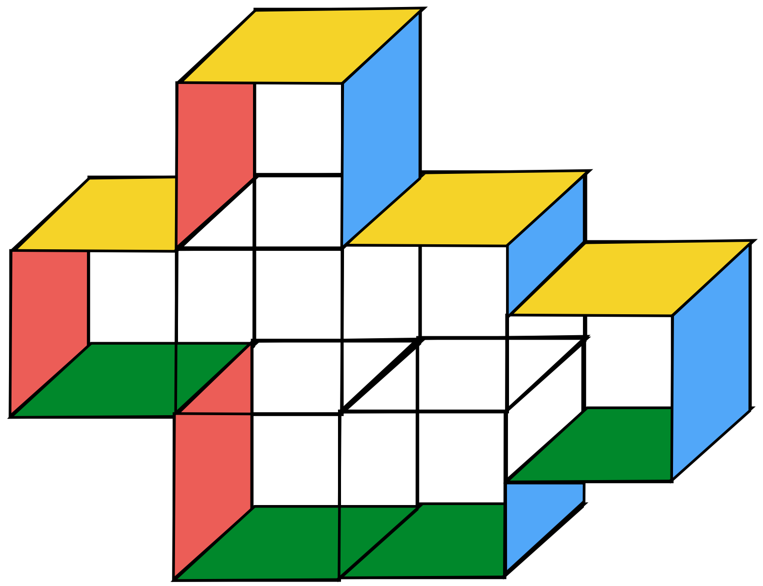
If we sum over the volume elements, this results in a sum over the surface elements!

Note: if two elementary surfaces touch, their $d\vec{a}$ vectors are in opposite directions!



Therefore the $d\vec{a}$ vectors cancel whenever there are two surfaces in touch





Thus the sum over surface elements gives the overall bounding surface!

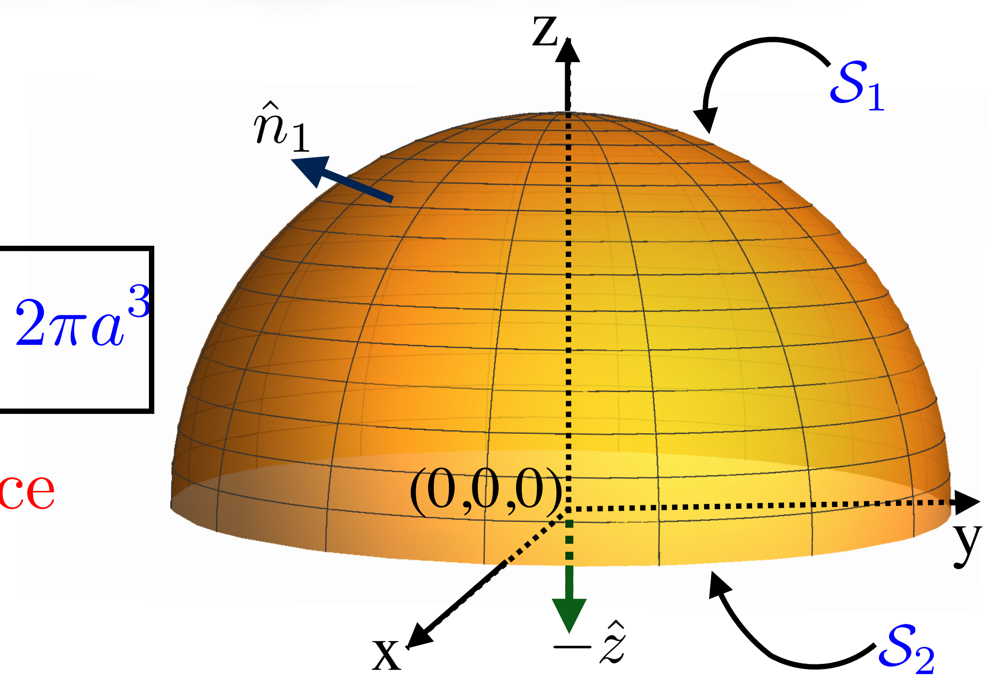
$$\int_{\mathcal{V}} \vec{\nabla}.\vec{F}\, d\tau = \oint_{\text{Surface of } \mathcal{V}} \vec{F}.d\vec{a}$$

Seems reasonable! Because, the “boundary” of a line are its endpoints, and boundary of a volume is a closed surface!

Example

Verify the divergence theorem when $\vec{F} = x\hat{x} + y\hat{y} + z\hat{z}$ and $\mathcal{S}$ is the surface composed of the upper half of the sphere of radius a and centred at the origin, together with the circular disc in xy-plane centred at the origin and of radius a.

$$\vec{\nabla}.\vec{F} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) = 1 + 1 + 1 = 3$$



$$\therefore \int_{\mathcal{V}} \vec{\nabla}.\vec{F}\, d\tau = 3.(\text{vol. of hemisphere}) = 3\,\frac{2}{3}\pi a^3 = 2\pi a^3$$

To check the result, we need to calculate the surface integral of $\vec{F}$ over the closed surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$.

Normal vector on $\mathcal{S}_1$ (hemisphere) : $\hat{n}_1 = \dfrac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{x\hat{x} + y\hat{y} + z\hat{z}}{a}$

Normal vector on $\mathcal{S}_2$ (disc at the base): $\hat{n}_2 = -\hat{z}$.

Surface integral for the flux through $\mathcal{S}_1 + \mathcal{S}_2$:

$$\oint_{\mathcal{S}_1+\mathcal{S}_2} \vec{F}.d\vec{a} = \int_{\mathcal{S}_1} \frac{x^2 + y^2 + z^2}{a} da + \underbrace{\int_{\mathcal{S}_2} (-z)da}_{= 0 \text{ because } z = 0 \text{ on } \mathcal{S}_2} = \int_{\mathcal{S}_1} a da$$

So, the value of the surface integral is $a(\text{area of } \mathcal{S}_1) = a(2\pi a^2) = 2\pi a^3$.

Surface of S_1 : $\phi(x, y, z) \equiv x^2 + y^2 + z^2 = a^2$ (const.)

$$\therefore \hat{n}_1 = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2x\,\hat{x} + 2y\,\hat{y} + 2z\,\hat{z}}{2\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\,\hat{x} + y\,\hat{y} + z\,\hat{z}}{a}$$

Example

Evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, where $\mathbf{F} = 4xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$ and S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

By the divergence theorem, the required integral is equal to

$$\iiint_V \nabla\cdot\mathbf{F}\, dV = \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_V (4z - y)\, dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z-y)\, dz\, dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} 2z^2 - yz \Big|_{z=0}^{1} dy\, dx = \int_{x=0}^{1}\int_{y=0}^{1} (2-y)\, dy\, dx = \frac{3}{2}$$

Example

Evaluate $\iint_S \mathbf{r}\cdot\mathbf{n}\, dS$, where S is a closed surface.

By the divergence theorem,

$$\iint_S \mathbf{r}\cdot\mathbf{n}\, dS = \iiint_V \nabla\cdot\mathbf{r}\, dV$$

$$= \iiint_V \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\, dV$$

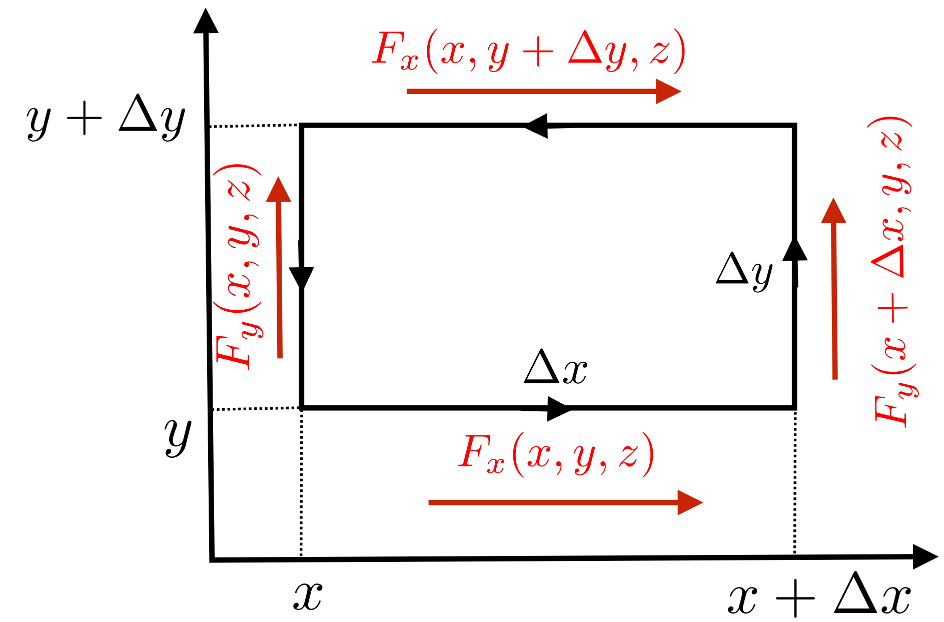
$$= \iiint_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) dV = 3\iiint_V dV = 3V$$

where V is the volume enclosed by S.

Fundamental theorem for Curl (Stokes' Theorem)

Let us find the circulation of a vector field $\vec{F}(x, y, z)$ around a closed curve C.

$$\oint_C \vec{F}.d\vec{r}$$



The fields in the x- direction at bottom and top are

$$F_x(x, y, z) \text{ and } F_x(x, y + \Delta y, z) = F_x + \frac{\partial F_x}{\partial y}\Delta y.$$

The fields in the y- direction at left and right are

$$F_y(x, y, z) \text{ and } F_y(x + \Delta x, y, z) = F_y + \frac{\partial F_y}{\partial x}\Delta x.$$

Summing around from bottom in anti-clockwise manner

$$\Delta C = \sum \vec{F}.\Delta\vec{r} = F_x(x, y, z)\Delta x + F_y(x + \Delta x, y, z)\Delta y - F_x(x, y + \Delta y, z)\Delta x - F_y(x, y, z)\Delta y$$

$$\sum \vec{F}.\Delta\vec{r} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\Delta x\Delta y = (\vec{\nabla} \times \vec{F}).\Delta x\Delta y\hat{z}$$

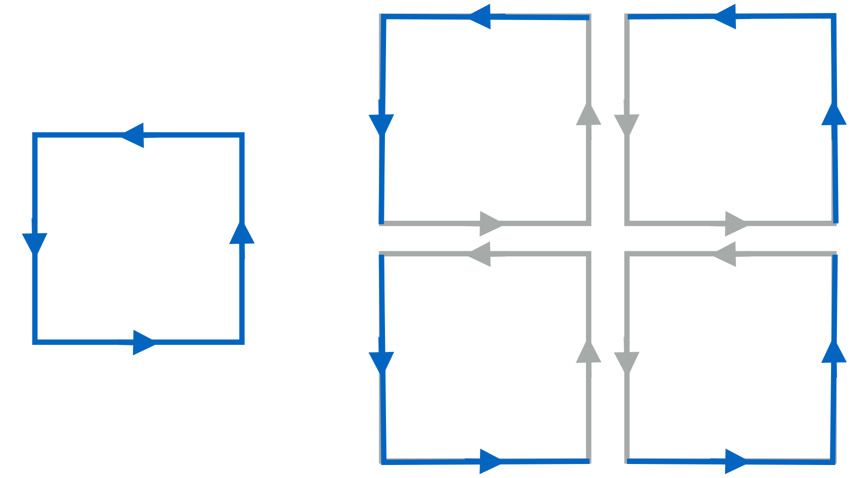
$$= (\vec{\nabla} \times \vec{F}).\Delta\vec{a}$$

This implies that curl can be defined as circulation per unit area...

Fundamental theorem for Curl (Stokes' Theorem)

Now, if we add these little elementary loops together, the internal line sections cancel out because the $d\vec{r}$'s are in opposite directions, except on the bounding line.

This gives the larger bounding contour.

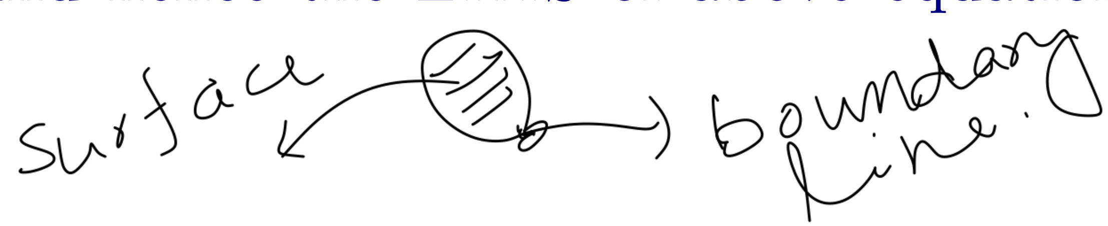


Stokes' Theorem: $$\oint_{\mathcal{C}} \vec{F}.d\vec{r} = \int_{\mathcal{S}} (\vec{\nabla} \times \vec{F}).d\vec{a}$$

Corollaries:

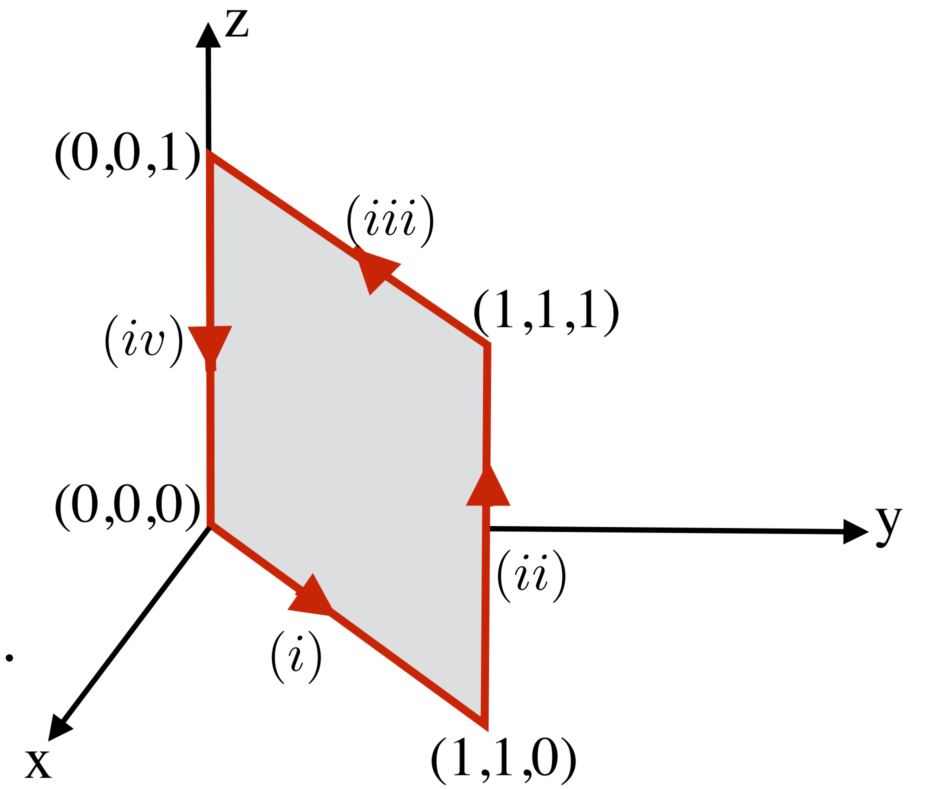
- $\int (\vec{\nabla} \times \vec{F}).d\vec{a}$ depends only on the boundary line, not on the particular surface used.

- $\oint (\vec{\nabla} \times \vec{F}).d\vec{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point and hence the L.H.S of above equation vanishes.



Example

Verify Stokes' theorem when $\mathcal{S}$ is the rectangle with vertices at $(0,0,0)$, $(1,1,0)$, $(0,0,1)$, and $(1,1,1)$, and $\vec{F} = yz\hat{x} + xz\hat{y} + xy\hat{z}$.



Direct Method:

Line integral $\oint_C \vec{F}.d\vec{r} = \oint_C yzdx + xzdy + xydz$ over path $(i) + (ii) + (iii) + (iv)$:

$\int_{(i)} \vec{F}.d\vec{r} = 0$, since $z = dz = 0$ on (i).

$\int_{(ii)} \vec{F}.d\vec{r} = \int_0^1 1.1dz = 1$, since $x = 1, y = 1$, $dx = 0 = dy$.

$\int_{(iii)} \vec{F}.d\vec{r} = \int_{(iii)} ydx + xdy = \int_1^0 xdx + xdx = -1$, since $y = x$, $z = 1, dz = 0$.

$\int_{(iv)} \vec{F}.d\vec{r} = 0$, since $x = 0,\ y = 0$, on (iv).

$$\oint_C \vec{F}.d\vec{r} = \left(\int_{(i)} + \int_{(ii)} + \int_{(iii)} + \int_{(iv)}\right) \vec{F}.d\vec{r} = 0$$

By Stokes' Theorem:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \hat{x}(x - x) + \hat{y}(y - y) + \hat{z}(z - z) = 0$$

$$\therefore \int_{\mathcal{S}} (\vec{\nabla} \times \vec{F}).d\vec{a} = 0$$

Example

Verify Stokes' theorem for $\mathbf{A} = (2x - y)\mathbf{i} - yz^2\,\mathbf{j} - y^2 z\,\mathbf{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

The boundary C of S is a circle in the xy plane of radius one and center at the origin. Let $x = \cos t$, $y = \sin t$, $z = 0$, $0 \leqq t < 2\pi$ be parametric equations of C. Then

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \oint_C (2x - y)\,dx - yz^2\,dy - y^2 z\,dz$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt = \pi$$

Also,

$$\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \mathbf{k}$$

Then

$$\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = \iint_S \mathbf{k}\cdot\mathbf{n}\,dS = \iint_R dx\,dy$$

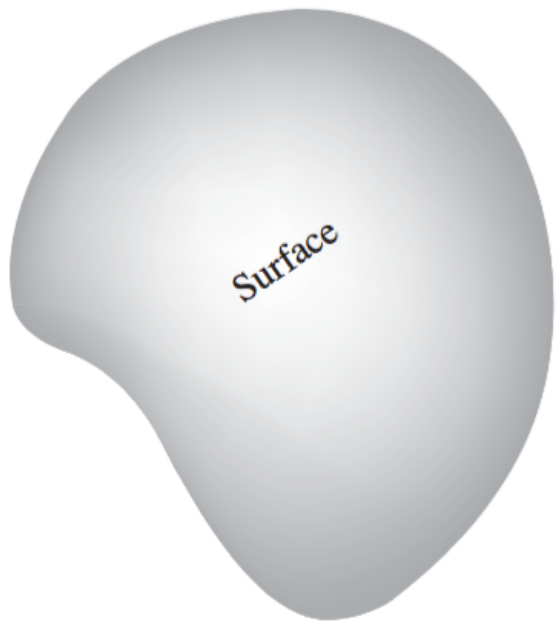
since $\mathbf{n}\cdot\mathbf{k}\,dS = dx\,dy$ and R is the projection of S on the xy plane. This last integral equals

$$\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1 \int_0^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1 \sqrt{1-x^2}\,dx = \pi$$

and Stokes' theorem is verified.

What did we learn today

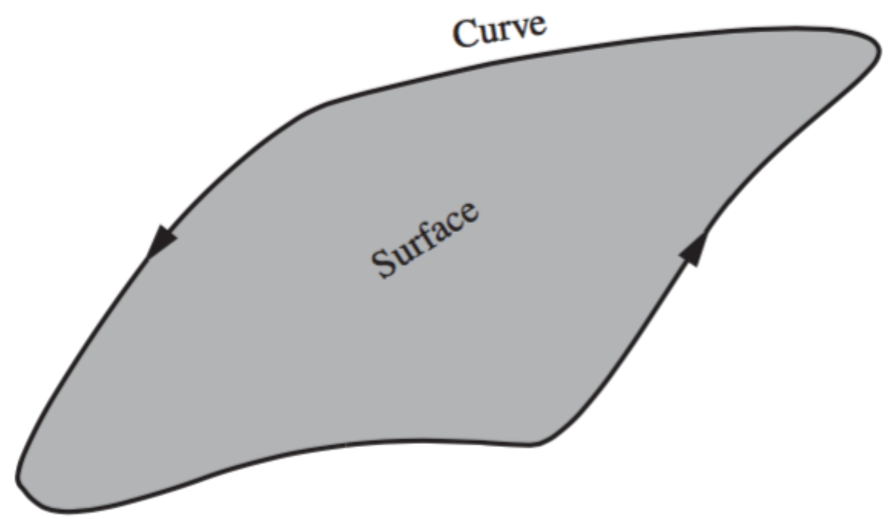
GAUSS



Surface encloses volume

$$\int_{\mathcal{S}} \vec{F}.d\vec{a} = \int_{\mathcal{V}} \vec{\nabla}.\vec{F} d\tau$$

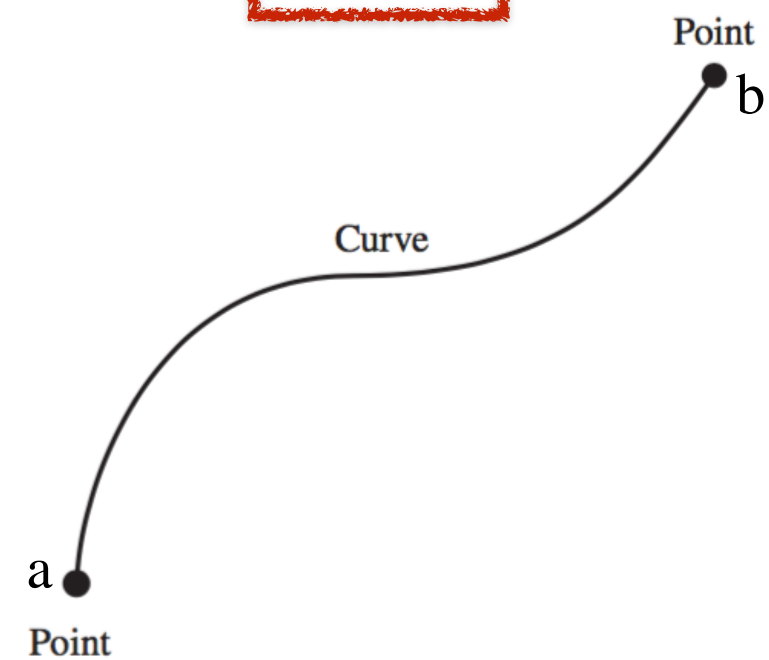
STOKES



Curve encloses surface

$$\oint_{\mathcal{C}} \vec{F}.d\vec{r} = \int_{\mathcal{S}} (\vec{\nabla} \times \vec{F}).d\vec{a}$$

GRAD



Points enclose curve

$$\int_{\mathcal{C}} \vec{\nabla}\phi.d\vec{r} = \phi(b) - \phi(a)$$

Remember:

In Cartesian Coordinates, with $\vec{\nabla} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)$

$$\vec{\nabla}.\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\vec{\nabla}\phi = \hat{x}\frac{\partial \phi}{\partial x} + \hat{y}\frac{\partial \phi}{\partial y} + \hat{z}\frac{\partial \phi}{\partial z}$$