Physics II: Electromagnetism

PH 102

Lecture 4

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January-May 2020

Orthogonal Curvilinear Coordinate System

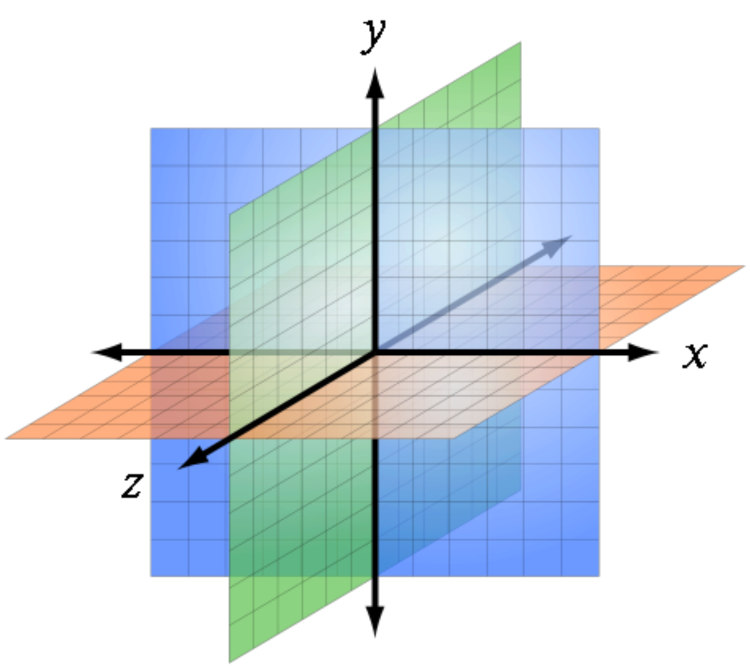
Before dealing with more examples of line, surface and volume integrals, it is better to understand how to convert an integral from one set of coordinates to another

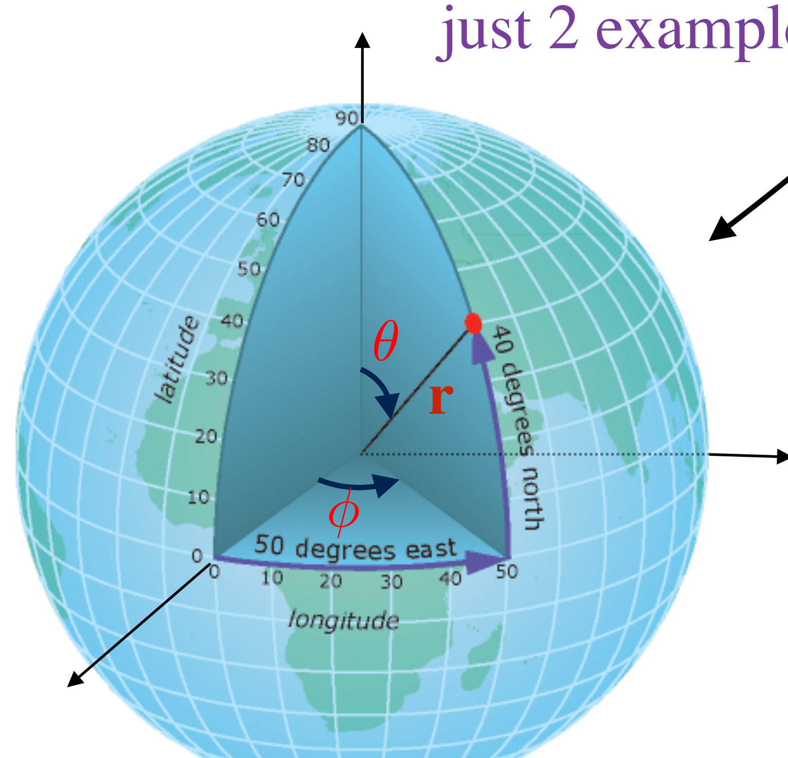
Why different set of coordinates are necessary?

> In Physics, symmetry plays a big role and often the symmetry of a problem screams at you to change the coordinate system to another one where the problem becomes much easier to handle

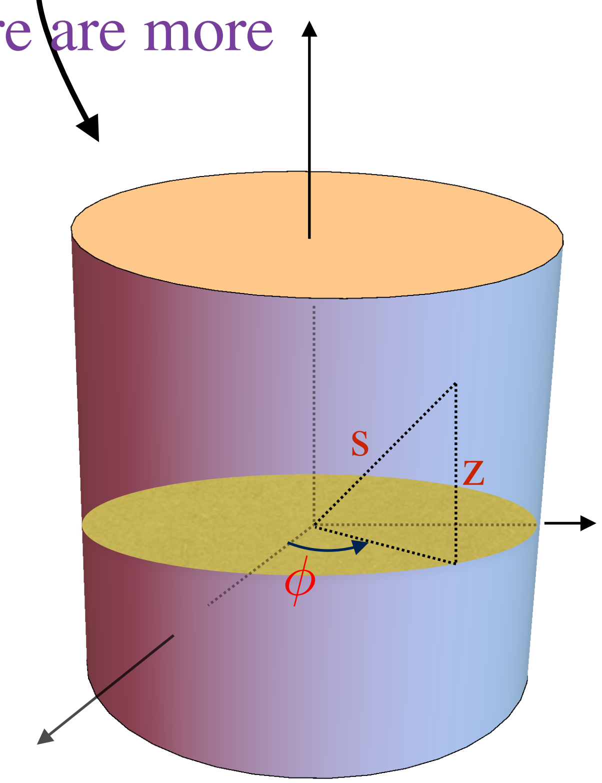
- Likely to be Plane (Cartesian), Spherical or Cylindrical polar coordinates
- But can be something more general like (u_1, u_2, u_3) $\longrightarrow$ Curvilinear coordinates

just 2 examples, there are more





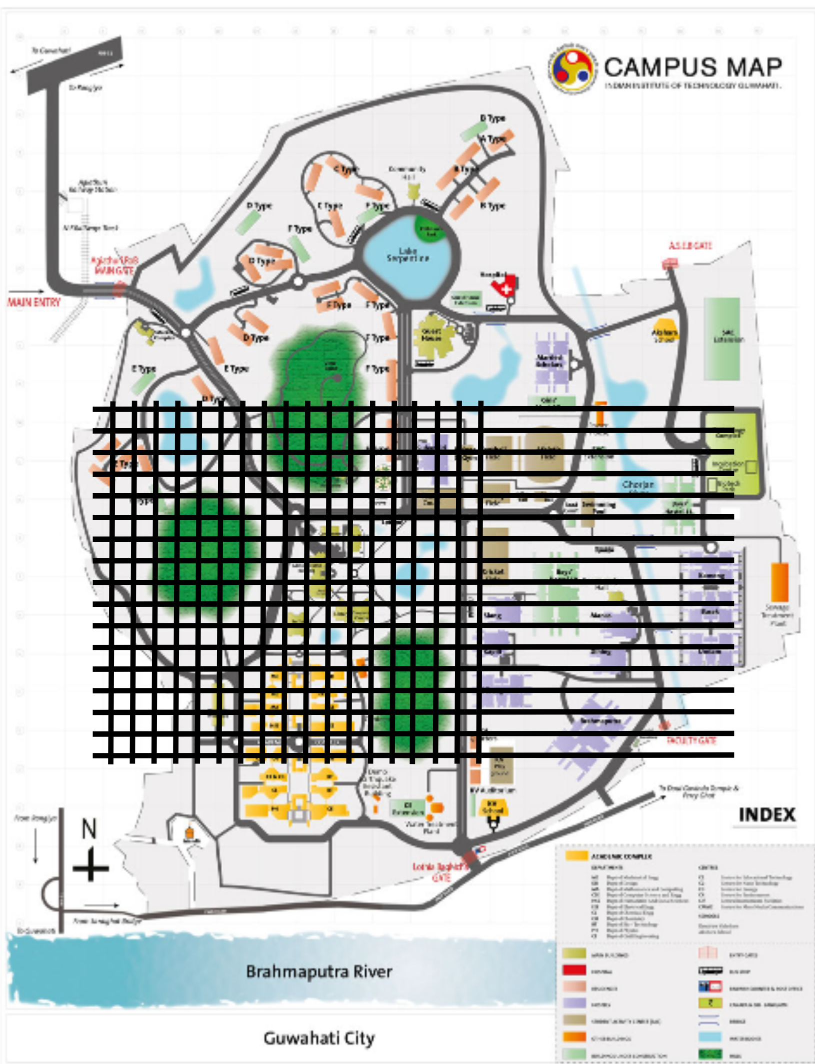
Spherical polar



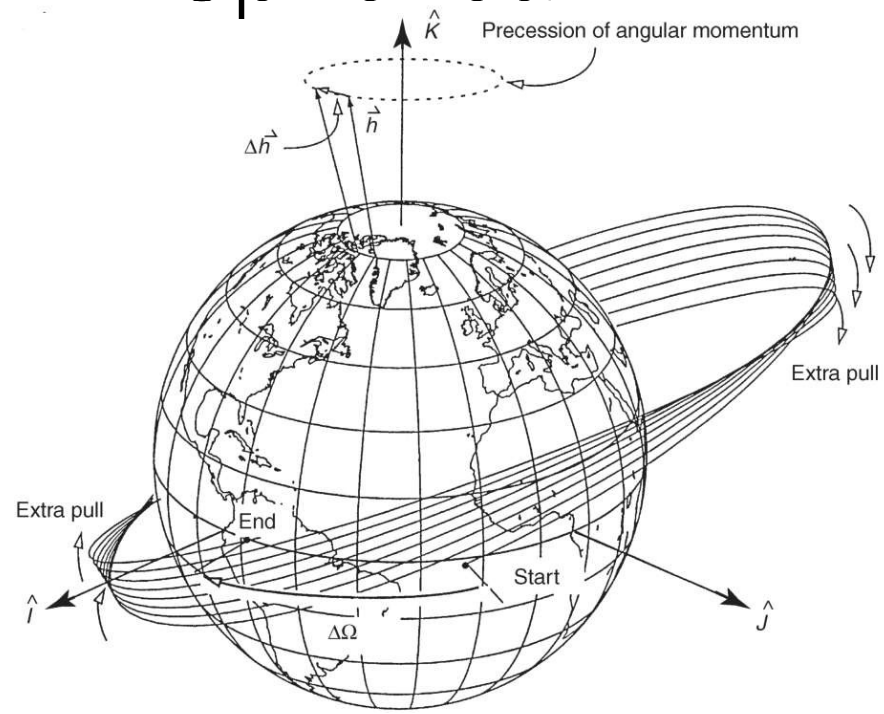
Cylindrical polar

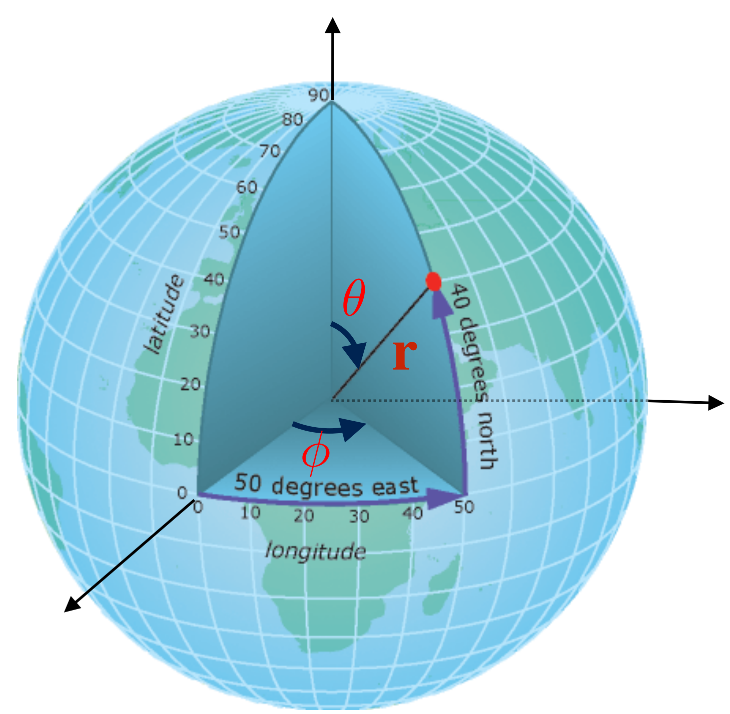
Applications

Cartesian

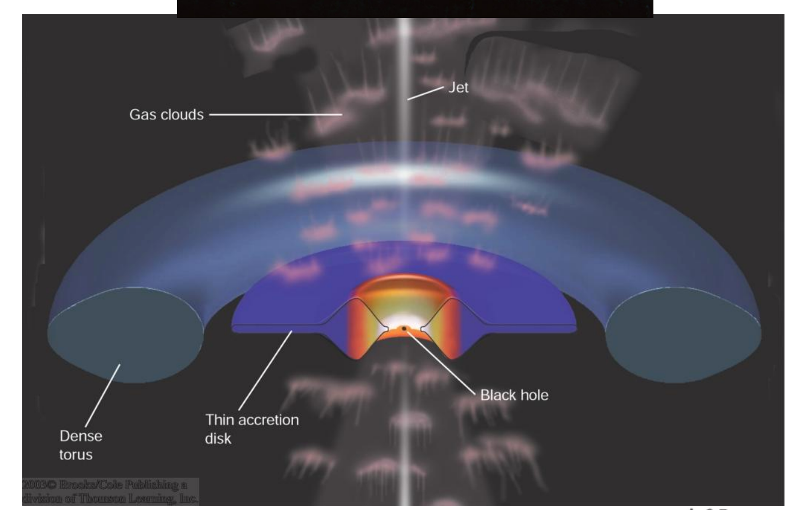


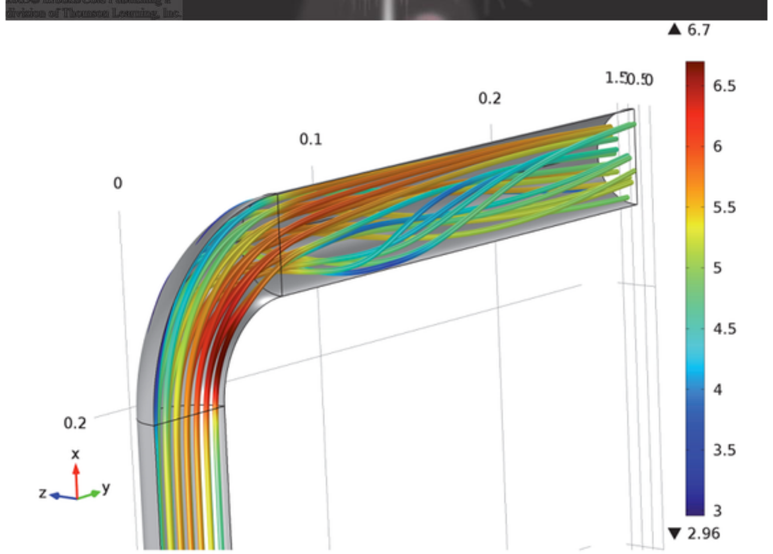
Spherical





Cylindrical

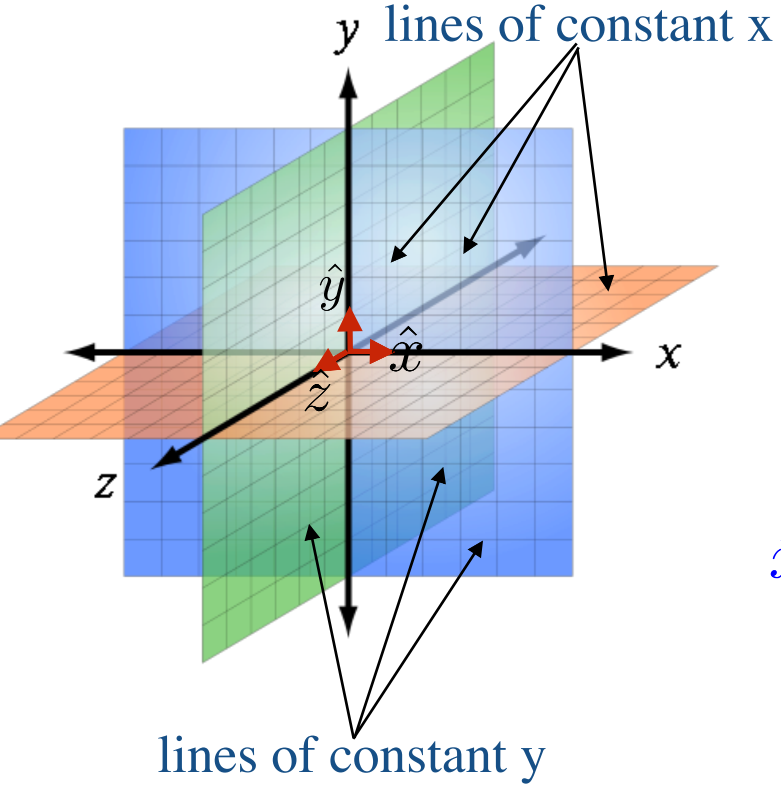




Recap: Cartesian Coordinate

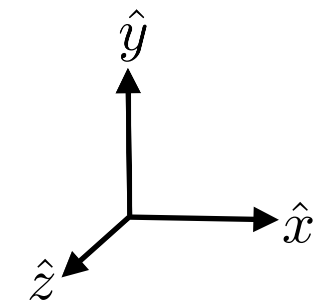
Recall Cartesian: $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ and $d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$, where $\hat{x}, \hat{y}, \hat{z}$ are constant unit vectors

$\left(d\vec{r} = \frac{\partial \vec{r}}{\partial x}dx + \frac{\partial \vec{r}}{\partial y}dy + \frac{\partial \vec{r}}{\partial z}dz\right)$ * unit vectors.



Length scales properly match: both LHS and RHS has the dimension of length: $|d\vec{r}| = \sqrt{dx^2 + dy^2 + dz^2}$

Unit vectors in Cartesian Coordinates:



$\hat{x}, \hat{y}, \hat{z}$: constant in direction (direction of increase of x, y and z)
constant in magnitude (norm=1)

Orthogonality: $\hat{x}_i.\hat{x}_j = \delta_{ij}$
Remember also: $\hat{x}_i \times \hat{x}_j = \epsilon_{ijk}\hat{x}_k$.

Suppose we want to go to curvilinear coordinates from Cartesian: $(x, y, z) \rightarrow (u_1, u_2, u_3)$

Bad News: unlike Cartesian, length scales are screwed up!

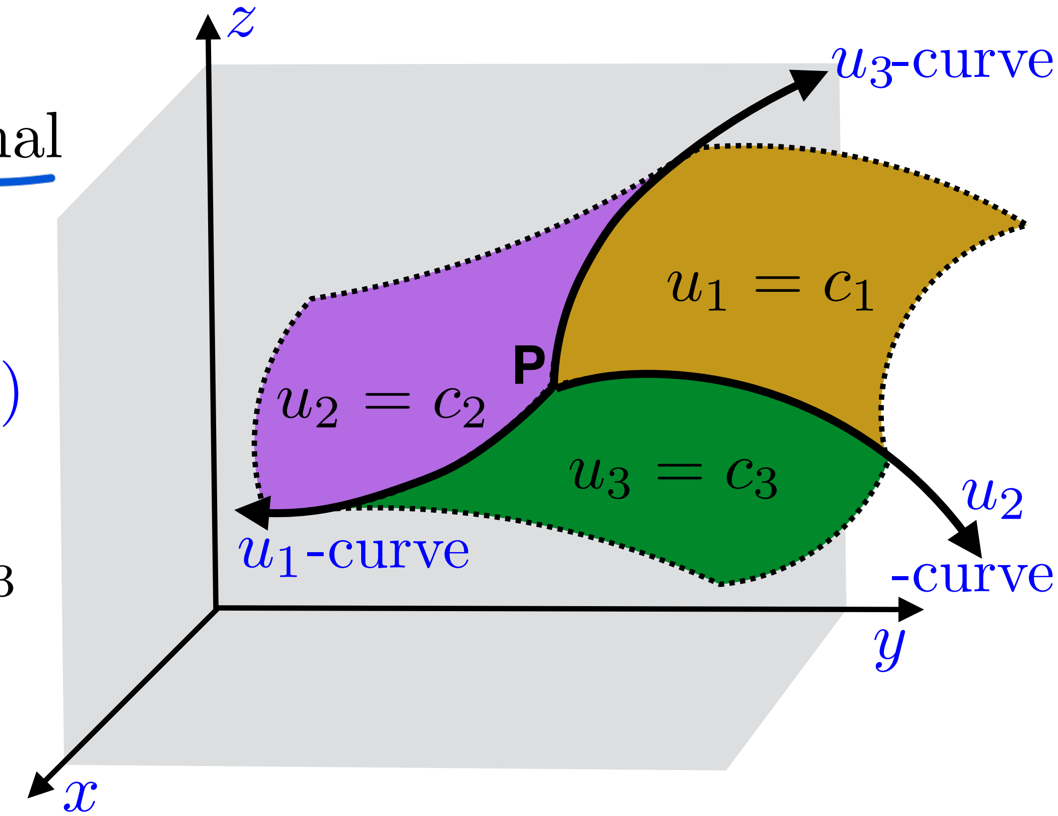
$$\begin{aligned} \vec{r} &\neq u_1\hat{u}_1 + u_2\hat{u}_2 + u_3\hat{u}_3 \\ d\vec{r} &\neq du_1\hat{u}_1 + du_2\hat{u}_2 + du_3\hat{u}_3 \\ |d\vec{r}| &\neq \sqrt{du_1^2 + du_2^2 + du_3^2} \end{aligned}$$

Think about $u_1 = r, u_2 = \theta, u_3 = \phi$, then the LHS has dimension of length, but RHS does not have the proper dimension.

From Cartesian to Curvilinear: Transformations

Consider the position vector at some point P in space. In Cartesian coordinates:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$



Now assume, at this point, we have another orthogonal coordinate system (u_1, u_2, u_3), such that

$$x = x(u_1, u_2, u_3), \;\; y = y(u_1, u_2, u_3), \;\; z = z(u_1, u_2, u_3)$$

Suppose, above eqns can be solved for u_1, u_2 and u_3 in terms of x, y, z:

$$u_1 = u_1(x, y, z), \;\; u_2 = u_2(x, y, z), \;\; u_3 = u_3(x, y, z)$$

Given a point P with Cartesian coordinates (x, y, z), we can associate a unique set of coordinates (u_1, u_2, u_3) called Curvilinear Coordinates of P.

The surfaces $u_1 = c_1$, $u_2 = c_2$ and $u_3 = c_3$ where c_1, c_2, c_3 are constants $\Longrightarrow$ Coordinate Surfaces

Each pair of these surfaces intersect at Coordinate Curves/lines

If Coordinate surfaces intersect at right angles $\Longrightarrow$ Orthogonal Curvilinear

From Cartesian to Curvilinear: unit vectors

We have just seen $x = x(u_1, u_2, u_3),\ y = y(u_1, u_2, u_3),\ z = z(u_1, u_2, u_3)$

Therefore $\vec{r} = x(u_1, u_2, u_3)\hat{x} + y(u_1, u_2, u_3)\hat{y} + z(u_1, u_2, u_3)\hat{z}$ ...and $d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$.

In order to define vector operators in this new coordinate system, we need to determine how the position vector changes with a change in this new coordinate system.

$$dx = \frac{\partial x}{\partial u_1}du_1 + \frac{\partial x}{\partial u_2}du_2 + \frac{\partial x}{\partial u_3}du_3;\ dy = \frac{\partial y}{\partial u_1}du_1 + \frac{\partial y}{\partial u_2}du_2 + \frac{\partial y}{\partial u_3}du_3;\ dz = \frac{\partial z}{\partial u_1}du_1 + \frac{\partial z}{\partial u_2}du_2 + \frac{\partial z}{\partial u_3}du_3$$

Hence,

$$d\vec{r} = \left(\frac{\partial x}{\partial u_1}du_1 + \frac{\partial x}{\partial u_2}du_2 + \frac{\partial x}{\partial u_3}du_3\right)\hat{x} + \left(\frac{\partial y}{\partial u_1}du_1 + \frac{\partial y}{\partial u_2}du_2 + \frac{\partial y}{\partial u_3}du_3\right)\hat{y} + \left(\frac{\partial z}{\partial u_1}du_1 + \frac{\partial z}{\partial u_2}du_2 + \frac{\partial z}{\partial u_3}du_3\right)$$

$$= \underbrace{\boxed{\left(\frac{\partial x}{\partial u_1}\hat{x} + \frac{\partial y}{\partial u_1}\hat{y} + \frac{\partial z}{\partial u_1}\hat{z}\right)du_1}}_{\frac{\partial \vec{r}}{\partial u_1}du_1} + \underbrace{\boxed{\left(\frac{\partial x}{\partial u_2}\hat{x} + \frac{\partial y}{\partial u_2}\hat{y} + \frac{\partial z}{\partial u_2}\hat{z}\right)du_2}}_{\frac{\partial \vec{r}}{\partial u_2}du_2} + \underbrace{\boxed{\left(\frac{\partial x}{\partial u_3}\hat{x} + \frac{\partial y}{\partial u_3}\hat{y} + \frac{\partial z}{\partial u_3}\hat{z}\right)du_3}}_{\frac{\partial \vec{r}}{\partial u_3}du_3}$$

$$= h_1\hat{e}_1 du_1 + h_2\hat{e}_2 du_2 + h_3\hat{e}_3 du_3$$

where $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are unit vectors in the direction of increasing u_1, u_2, u_3.

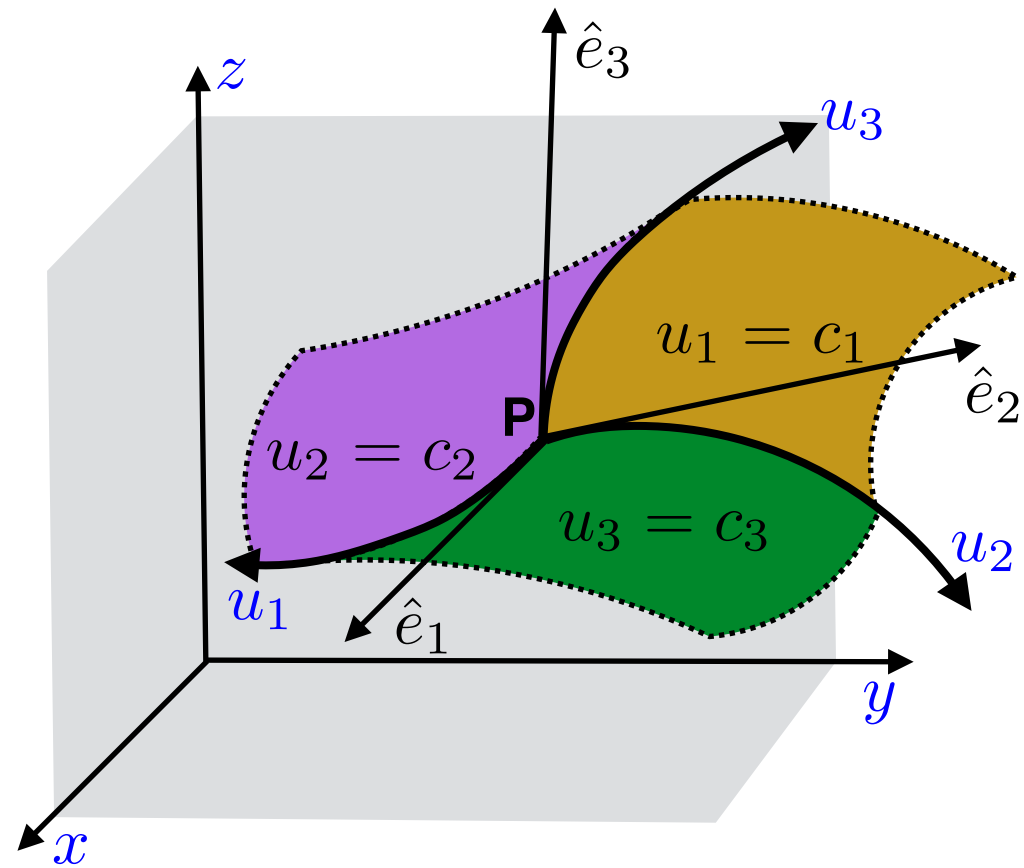
h_1, h_2, h_3 are called Scale Factors.

From Cartesian to Curvilinear: unit vectors

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

$$= h_1 \hat{e}_1 du_1 + h_2 \hat{e}_2 du_2 + h_3 \hat{e}_3 du_3$$

$$\therefore h_1 \hat{e}_1 = \frac{\partial \vec{r}}{\partial u_1};\ h_2 \hat{e}_2 = \frac{\partial \vec{r}}{\partial u_2};\ h_3 \hat{e}_3 = \frac{\partial \vec{r}}{\partial u_3}$$



Note that a tangent vector to u_1 curve at P (for which u_2, u_3 are constants) is $\frac{\partial \vec{r}}{\partial u_1}$. Then a unit tangent vector in this direction is $\hat{e_1} = \frac{\partial \vec{r}}{\partial u_1} / |\frac{\partial \vec{r}}{\partial u_1}|$.

Similarly $\hat{e}_2 = \frac{\partial \vec{r}}{\partial u_2} / |\frac{\partial \vec{r}}{\partial u_2}|$ and $\hat{e}_3 = \frac{\partial \vec{r}}{\partial u_3} / |\frac{\partial \vec{r}}{\partial u_3}|$

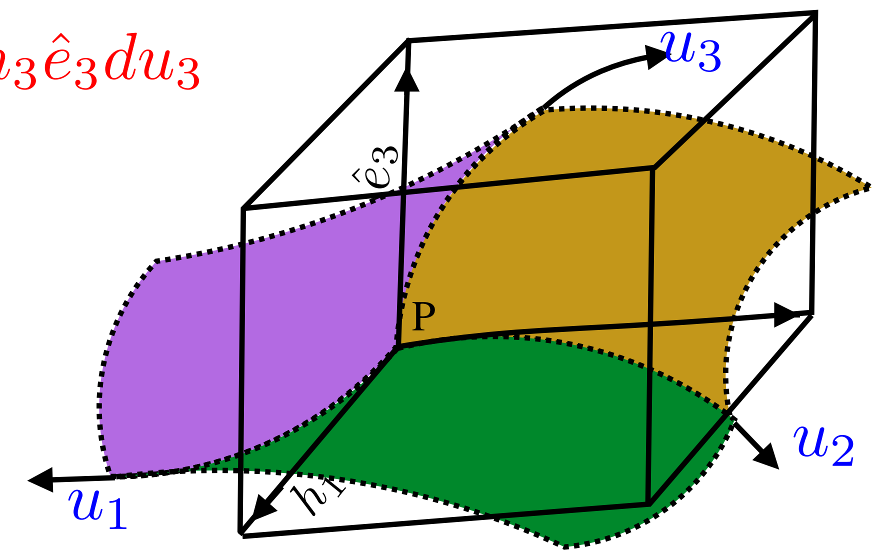
The scale factors are therefore: $h_1 = |\frac{\partial \vec{r}}{\partial u_1}|;\ h_2 = |\frac{\partial \vec{r}}{\partial u_2}|;\ h_3 = |\frac{\partial \vec{r}}{\partial u_3}|$

: relate the actual displacement in a given coordinate direction to the change of that coordinate.

Unit vectors here are analogous to the unit vectors in cartesian coordinates but are unlike them in that they may change directions from point to point.

In a orthogonal curvilinear coordinate the unit vectors are orthogonal (perpendicular) to each other.

Arc length, Volume element etc…



$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3 = h_1 \hat{e}_1 du_1 + h_2 \hat{e}_2 du_2 + h_3 \hat{e}_3 du_3$$

Differential of arc length ds: $ds^2 = d\vec{r}.d\vec{r}$ (why?)

Since $\hat{e}_i.\hat{e}_j = \delta_{ij}$, $\quad ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2.$

Along a u_1 curve, u_2 and u_3 are constants so that $d\vec{r} = h_1 du_1 \hat{e}_1$.

Hence, the differential arc length ds_1 along u_1 curve at P is $h_1 du_1$.

Similarly $ds_2 = h_2 du_2$ and $ds_3 = h_3 du_3$ along u_2 and u_3 at P.

Volume element:

Look at the parallelepiped formed out of the vectors $h_1 du_1 \hat{e}_1$, $h_2 du_2 \hat{e}_2$ and $h_3 du_3 \hat{e}_3$: the volume element is given by:

$$d\tau = |(h_1 du_1 \hat{e}_1).(h_2 du_2 \hat{e}_2) \times (h_3 du_3 \hat{e}_3)| = h_1 h_2 h_3 du_1 du_2 du_3,$$

since $|\hat{e}_1.(\hat{e}_2 \times \hat{e}_3)|$. $= |\hat{e}_1 . \hat{e}_1| = 1$

Gradient operator in Curvilinear coordinate

We have already seen that $d\vec{r} = h_1\hat{e}_1 du_1 + h_2\hat{e}_2 du_2 + h_3\hat{e}_3 du_3$

The scalar function T is now a function of curvilinear coordinates (u_1, u_2, u_3).

Therefore, $dT(u_1, u_2, u_3) = \vec{\nabla}T(u_1, u_2, u_3).d\vec{r}$.

But, $dT(u_1, u_2, u_3) = \frac{\partial T}{\partial u_1}du_1 + \frac{\partial T}{\partial u_2}du_2 + \frac{\partial T}{\partial u_3}du_3$.

It follows that:

$$\vec{\nabla}T.(h_1\hat{e}_1 du_1 + h_2\hat{e}_2 du_2 + h_1\hat{e}_3 du_3) = \frac{\partial T}{\partial u_1}du_1 + \frac{\partial T}{\partial u_2}du_2 + \frac{\partial T}{\partial u_3}du_3$$

The only way it can be satisfied for independent du_1, du_2 and du_3 is when

$\vec{\nabla}T(u_1, u_2, u_3)$ is curvilinear coordinates:

$$\boxed{\vec{\nabla}T(u_1, u_2, u_3) = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{e}_3.}$$

Divergence, Curl and Laplacian in Curvilinear Coordinates

Proceeding in a similar manner, one can check, after a few lines of calculations:

Divergence: $$\vec{\nabla}.\vec{V} = \frac{1}{h_1 h_2 h_3}\Big(\frac{\partial(h_2 h_3 V_1)}{\partial u_1} + \frac{\partial(h_3 h_1 V_2)}{\partial u_2} + \frac{\partial(h_1 h_2 V_3)}{\partial u_3}\Big)$$

Curl: $$\vec{\nabla}\times\vec{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

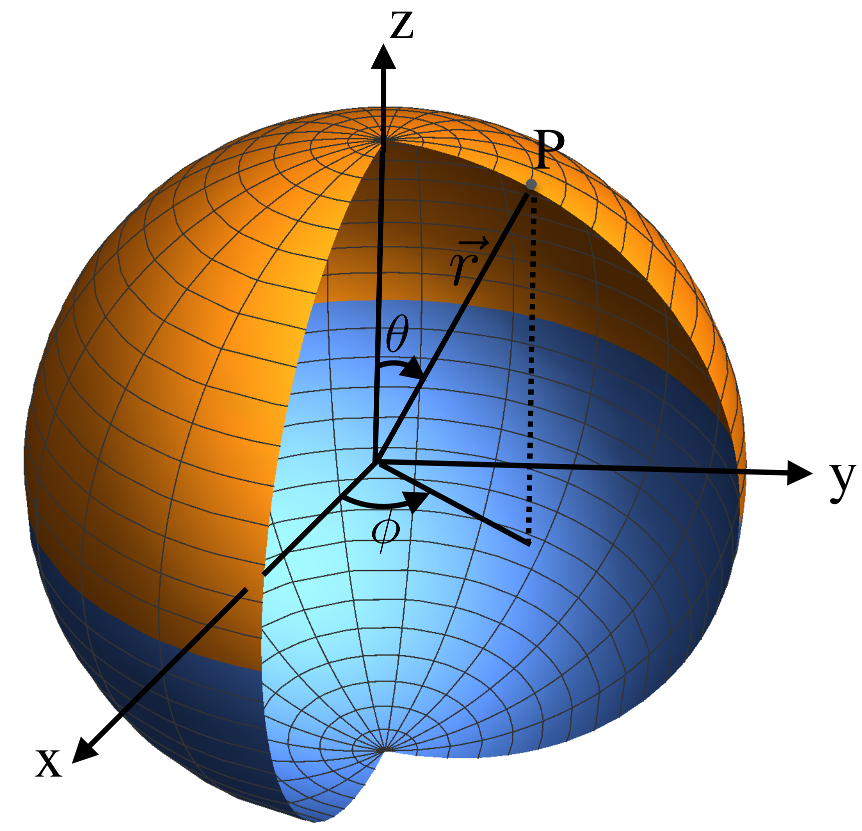
Laplacian: $$\nabla^2 \mathrm{T} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\Big(\frac{h_2 h_3}{h_1}\frac{\partial \mathrm{T}}{\partial u_1}\Big) + \frac{\partial}{\partial u_2}\Big(\frac{h_3 h_1}{h_2}\frac{\partial \mathrm{T}}{\partial u_2}\Big) + \frac{\partial}{\partial u_3}\Big(\frac{h_1 h_2}{h_3}\frac{\partial \mathrm{T}}{\partial u_3}\Big)\right]$$

Quick Check

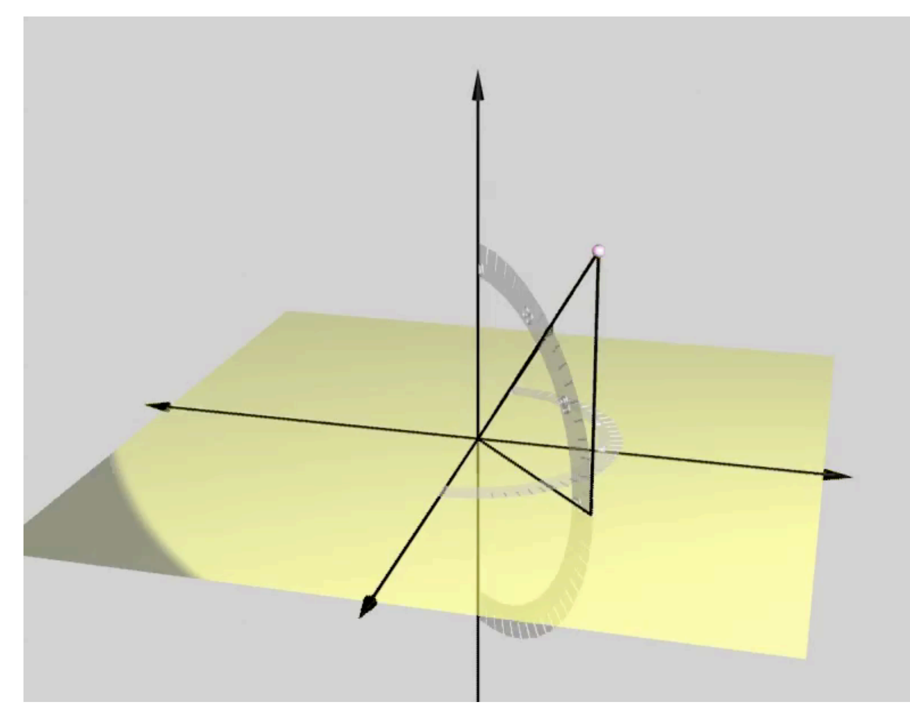
For Cartesian coordinates, $h_1 = h_2 = h_3 = 1$ and $\hat{e}_1 = \hat{x},\ \hat{e}_2 = \hat{y},\ \hat{e}_3 = \hat{z}$. This reduces the above expressions to the familiar expressions in Cartesian coordinate where (u_1, u_2, u_3) are replaced by (x, y, z).

Specific examples:
Spherical Polar and Cylindrical Polar

Spherical Polar Coordinates



- Cartesian coordinate of P: (x, y, z)
- Position vector of P: $\vec{r}$
- Length of $\vec{r}$: $r = |\vec{r}|$
- Polar angle (angle between z axis and $\vec{r}$): θ
- Azimuthal angle (angle between x axis and projection of $\vec{r}$ on xy plane): ϕ
- Spherical Polar Coordinate: $(r, \theta, \phi) \equiv (u_1, u_2, u_3)$



- Range of r: $0 \leq r < \infty$
- Range of θ: $0 \leq \theta \leq \pi$
- Range of ϕ: $0 \leq \phi < 2\pi$
- Transformations: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$
- $\vec{r} = r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta\hat{z}$
- Inverse transformations: $r = \sqrt{x^2+y^2+z^2}$, $\theta = \sin^{-1}\dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$, $\phi = \tan^{-1}\left(\dfrac{y}{x}\right)$

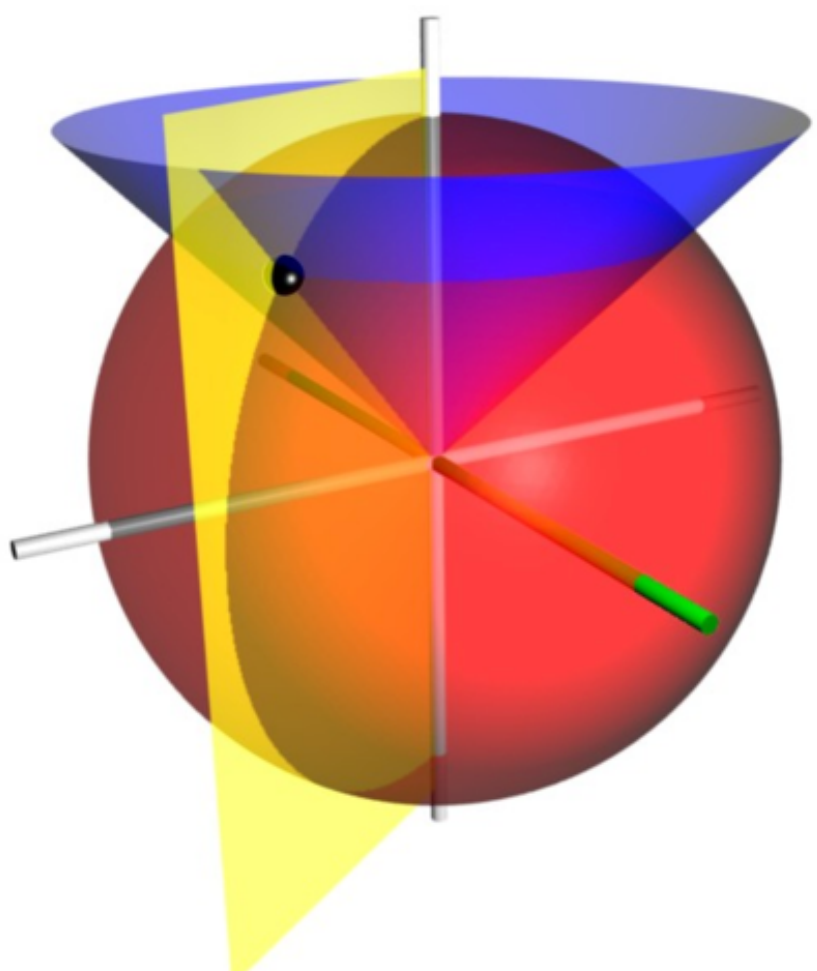
Spherical Polar Coordinates

Coordinate Surfaces:

Recall: coordinate surfaces were defined as surfaces obtained by keeping one of the coordinates (either u_1 or u_2 or u_3 constant) constant. Here $(u_1, u_2, u_3) = (r, \theta, \phi)$.

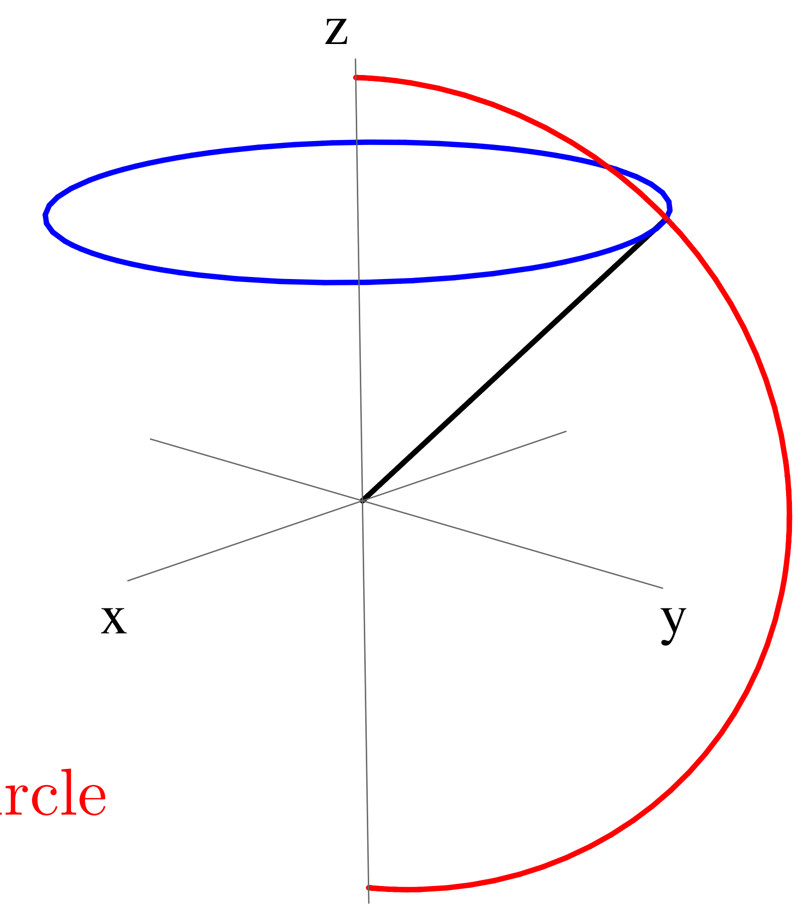
The coordinate surfaces are:

$$\begin{aligned} r &= c_1, && \text{spheres having centre at the origin} \\ \theta &= c_2, && \text{cones having vertex at origin (line if } c_2 = 0 \text{ or } \pi,\ xy \text{ plane if } c_2 = \pi/2) \\ \phi &= c_3, && \text{planes through z axis} \end{aligned}$$



Spherical Polar Coordinates

Coordinate Curves:



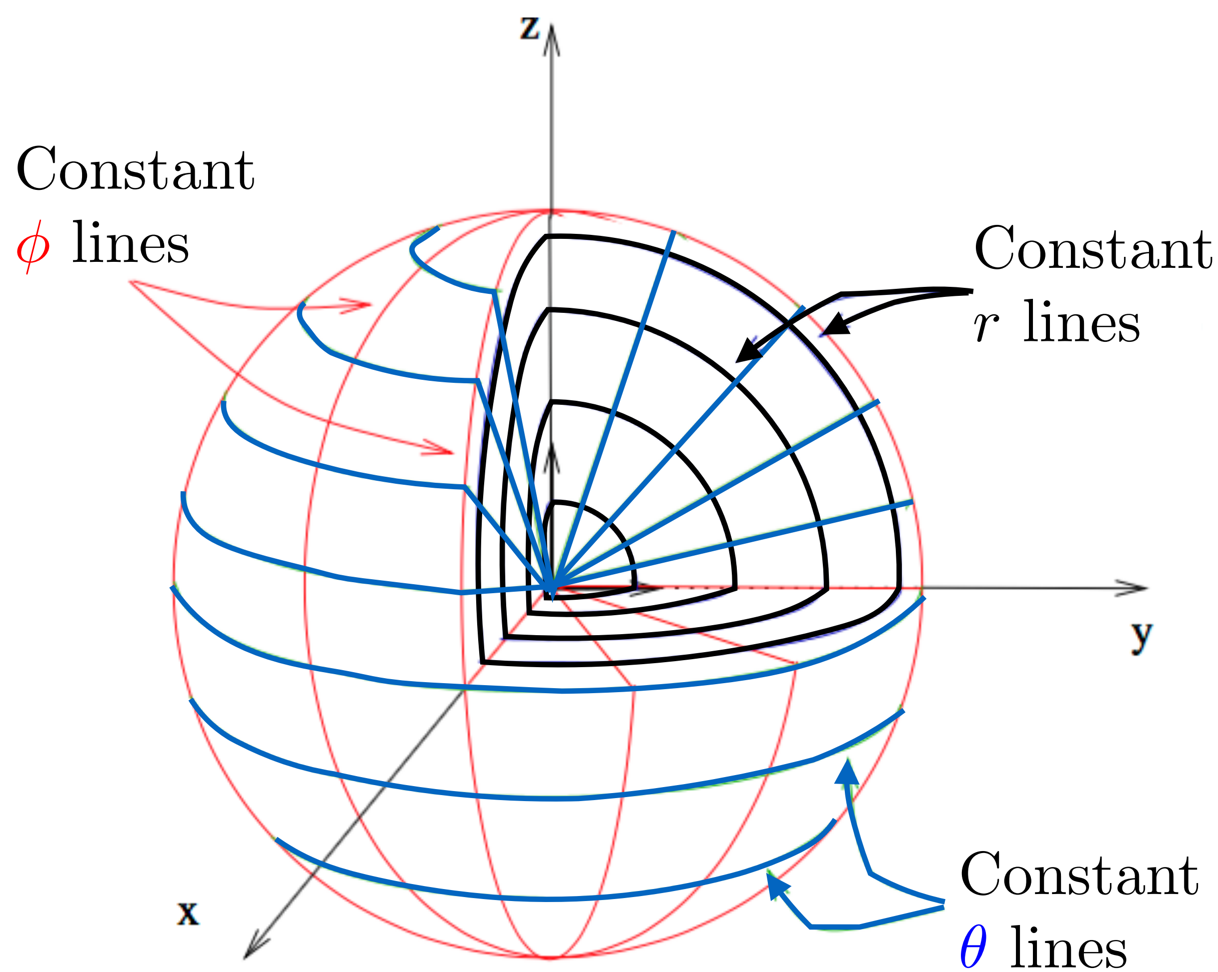
Recall: coordinate curves were obtained by keeping two coordinates fixed (intersection of $u_1 = c_1$ or $u_2 = c_2$ or $u_3 = c_3$ surfaces).

Intersection of $\mathrm{r} \;=\; c_1$ and $\theta = \mathrm{c}_2$ ($\phi -$ curve) is a circle

Intersection of $\mathrm{r} \;=\; c_1$ and $\phi = \mathrm{c}_3$ ($\theta -$ curve) is a semi circle

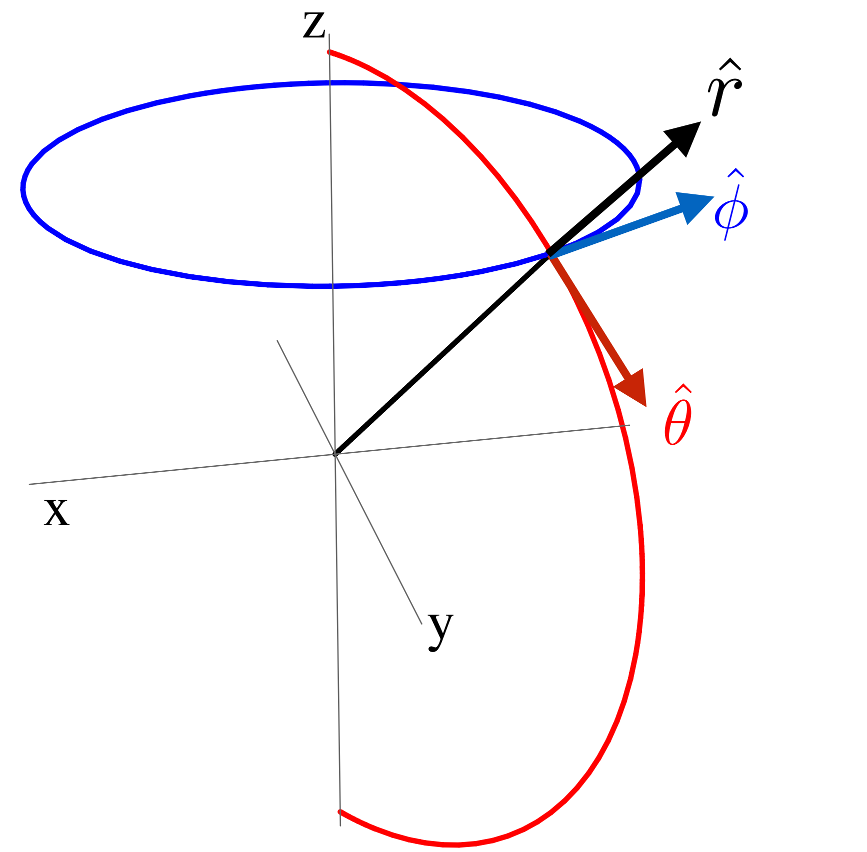
Intersection of $\theta \;=\; c_2$ and $\phi = \mathrm{c}_3$ (r $-$ curve) is a line

- Lines of constant ϕ : Longitude
- Lines of constant θ : Lattitude



Spherical Polar Coordinates: Unit vectors and Scale factors

$\vec{r} = r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta\hat{z}$



Recall that $\hat{e}_i = \frac{1}{h_i}\frac{\partial\vec{r}}{\partial u_i}$, where $h_i = |\frac{\partial\vec{r}}{\partial u_i}|$.

Hence $h_1 \equiv h_r = |\frac{\partial\vec{r}}{\partial r}| = 1$, $h_2 \equiv h_\theta = |\frac{\partial\vec{r}}{\partial\theta}| = r$, $h_3 \equiv h_\phi = |\frac{\partial\vec{r}}{\partial\phi}| = r\sin\theta$

Unit vectors:

$$\hat{e}_1 \equiv \hat{r} = \frac{\frac{\partial\vec{r}}{\partial r}}{\left|\frac{\partial\vec{r}}{\partial r}\right|} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$$

$$\hat{e}_2 \equiv \hat{\theta} = \frac{\frac{\partial\vec{r}}{\partial\theta}}{\left|\frac{\partial\vec{r}}{\partial\theta}\right|} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$$

$$\hat{e}_3 \equiv \hat{\phi} = \frac{\frac{\partial\vec{r}}{\partial\phi}}{\left|\frac{\partial\vec{r}}{\partial\phi}\right|} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

Check

- $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$
- $\hat{e}_i \times \hat{e}_j = \epsilon_{ijk}\hat{e}_k$

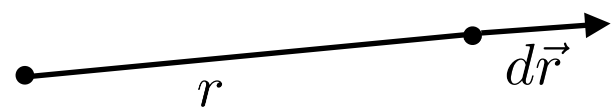
This shows that the unit vectors in spherical polar coordinates are dependent on position

The unit vectors $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are in the directions of increasing r, θ and ϕ respectively.

Spherical Polar: Line, Volume and Surface elements

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3 = h_1 \hat{e}_1 du_1 + h_2 \hat{e}_2 du_2 + h_3 \hat{e}_3 du_3$$

Therefore, for spherical polar $d\vec{r} = \hat{r} dr + r d\theta \hat{\theta} + r \sin\theta d\phi \hat{\phi} \neq dr \hat{r} + d\theta \hat{\theta} + d\phi \hat{\phi}$

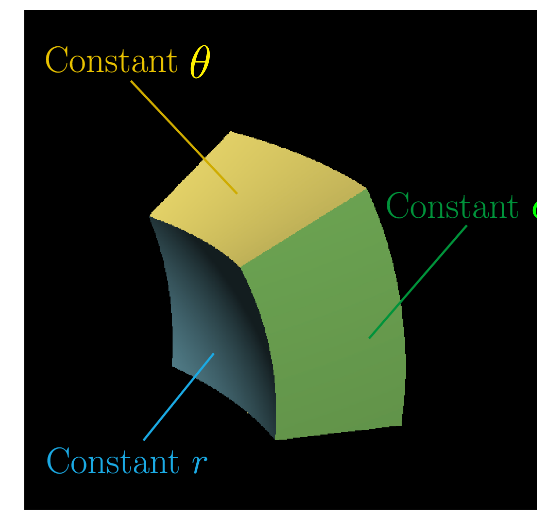


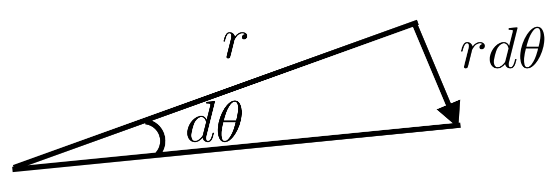
Scale factors take care of the length scale

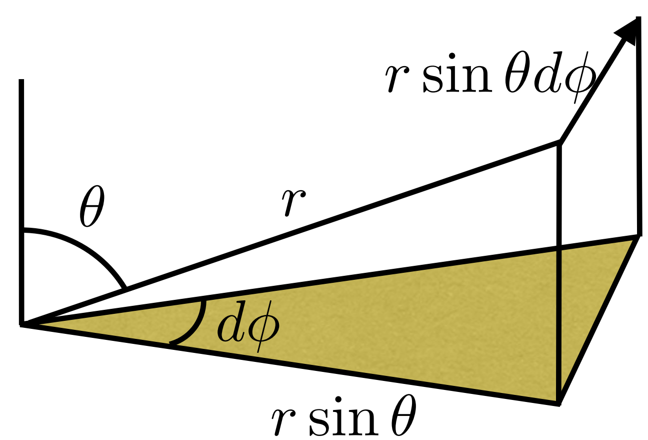
Volume element: $d\tau = h_r h_\theta h_\phi dr d\theta d\phi = r^2 \sin\theta dr d\theta d\phi.$

Surface element: No general expression. Depend on orientation of the surface:

$$\begin{aligned} d\vec{a}_r &= h_\theta h_\phi d\theta d\phi \hat{r} = r^2 \sin\theta d\theta d\phi \hat{r} \quad (r \text{ constant surface}) \\ d\vec{a}_\theta &= h_r h_\phi dr d\phi \hat{\theta} = r \sin\theta dr d\phi \hat{\theta} \quad (\theta \text{ constant surface}) \\ d\vec{a}_\phi &= h_r h_\theta dr d\theta \hat{\phi} = r dr d\theta \hat{\phi} \quad (\phi \text{ constant surface}) \end{aligned}$$







Find out the expressions for the gradient, divergence, curl and the Laplacian in the spherical polar coordinate using the general form in curvilinear coordinate.

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}.$$

Curl:

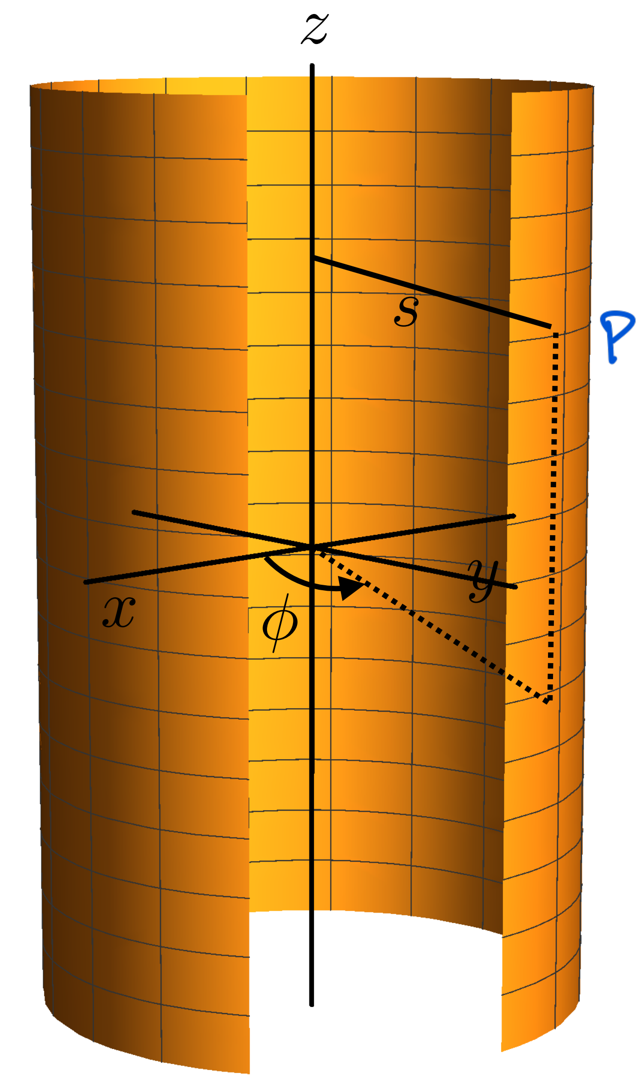
$$\begin{aligned}\nabla \times \mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}.\end{aligned}$$

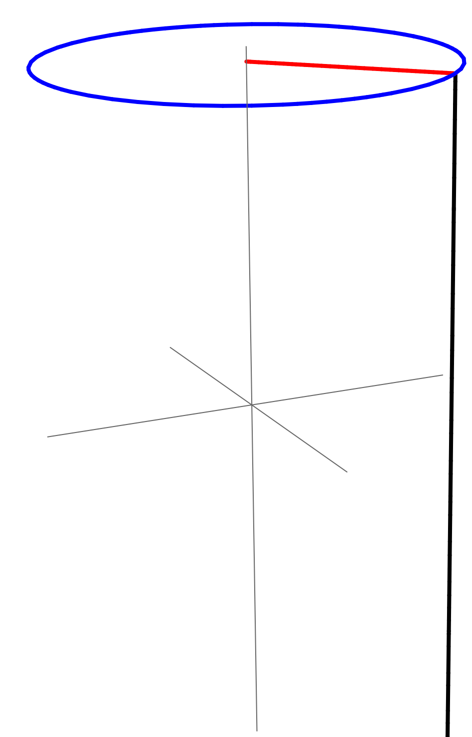
Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}.$$

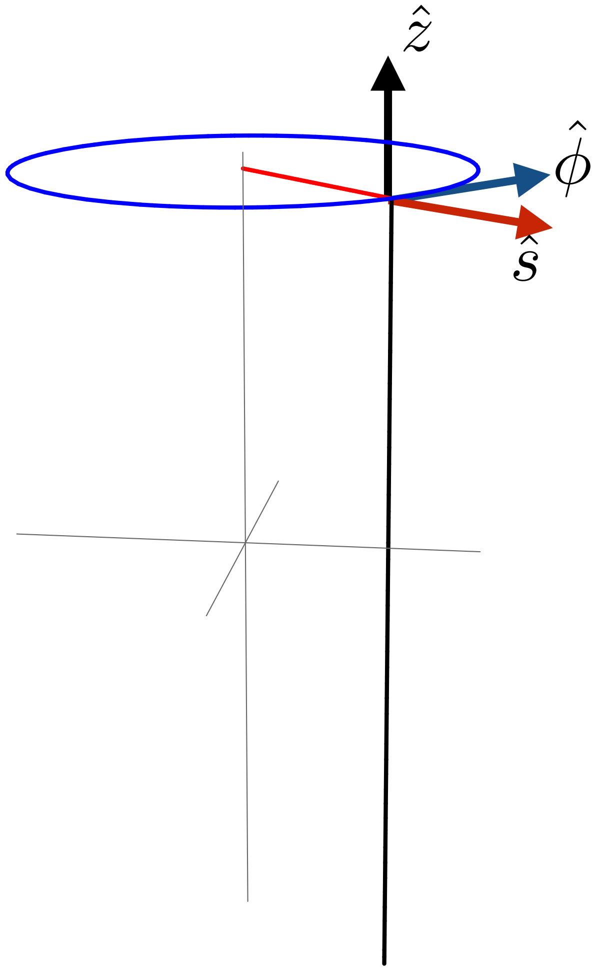
Cylindrical Polar Coordinates

- Cartesian coordinate of P: (x, y, z)
- Distance of P from z axis: s
- Height: z (same as Cartesian)
- Azimuthal angle: ϕ (same as spherical polar)
- Cylindrical Polar Coordinate: $(s, \phi, z) \equiv (u_1, u_2, u_3)$
- Range of s: $0 \leq s < \infty$
- Range of ϕ: $0 \leq \phi < 2\pi$
- Range of z: $-\infty < z < \infty$
- Transformations: $x = s\cos\phi,\ \ y = s\sin\phi,\ \ z = z$
- Inverse transformations: $s = \sqrt{x^2 + y^2},\ \ \phi = \tan^{-1}\left(\frac{y}{x}\right),\ \ z = z$
- Coordinate surfaces and curves: Find out!

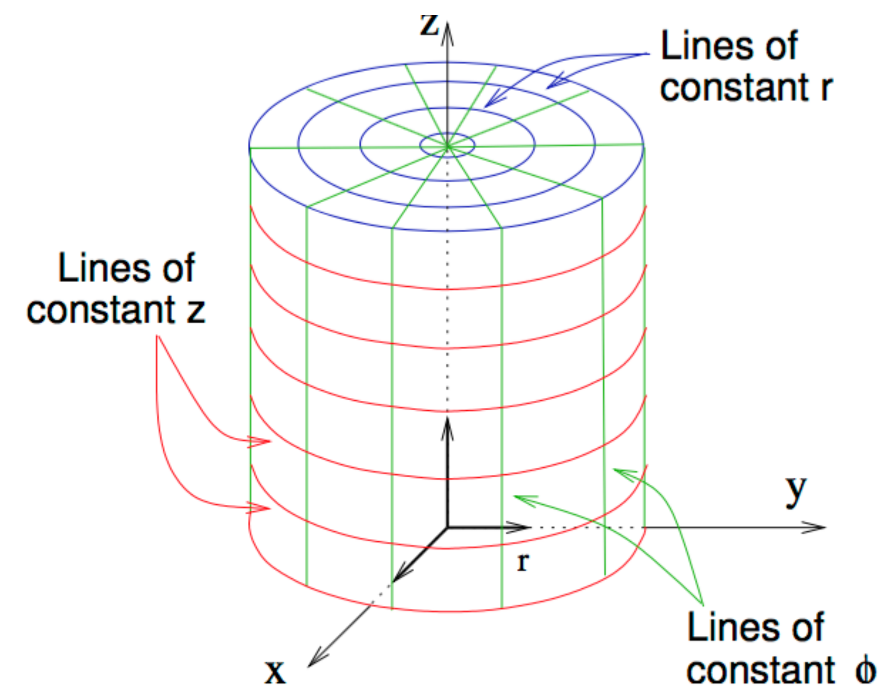




Cylindrical Polar Coordinates



As usual, the scale factors are given by:
$h_1 \equiv h_s = 1$, $h_2 \equiv h_\phi = s$, $h_3 \equiv h_z = 1$.



The unit vectors are:

$$\hat{s} = \cos\phi\hat{x} + \sin\phi\hat{y}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

$$\hat{z} = \hat{z}$$

Line element: $d\vec{r} = h_s\hat{s}ds + h_\phi\hat{\phi}d\phi + h_z\hat{z}dz = ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z}$

Surface element: $d\vec{a}_s = h_\phi h_z d\phi dz\hat{s} = sd\phi dz\hat{s}$ (for s constant surface)

Volume element: $d\tau = h_s h_\phi h_z ds d\phi dz = s ds d\phi dz$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\,\hat{\mathbf{z}}.$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}.$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}.$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}.$$

Example

Represent the vector $\mathbf{A} = z\mathbf{i} - 2x\mathbf{j} + y\mathbf{k}$ in cylindrical coordinates. Thus determine A_ρ, A_ϕ and A_z.

From Problem 3,

$$(1)\ \mathbf{e}_\rho = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j} \qquad (2)\ \mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j} \qquad (3)\ \mathbf{e}_z = \mathbf{k}$$

Solving (1) and (2) simultaneously,

$$\mathbf{i} = \cos\phi\,\mathbf{e}_\rho - \sin\phi\,\mathbf{e}_\phi, \qquad \mathbf{j} = \sin\phi\,\mathbf{e}_\rho + \cos\phi\,\mathbf{e}_\phi$$

Then

$$\begin{aligned}\mathbf{A} &= z\mathbf{i} - 2x\mathbf{j} + y\mathbf{k}\\ &= z(\cos\phi\,\mathbf{e}_\rho - \sin\phi\,\mathbf{e}_\phi) - 2\rho\cos\phi(\sin\phi\,\mathbf{e}_\rho + \cos\phi\,\mathbf{e}_\phi) + \rho\sin\phi\,\mathbf{e}_z\\ &= (z\cos\phi - 2\rho\cos\phi\sin\phi)\mathbf{e}_\rho - (z\sin\phi + 2\rho\cos^2\phi)\mathbf{e}_\phi + \rho\sin\phi\,\mathbf{e}_z\end{aligned}$$

and $A_\rho = z\cos\phi - 2\rho\cos\phi\sin\phi$, $A_\phi = -z\sin\phi - 2\rho\cos^2\phi$, $A_z = \rho\sin\phi$.

Example

Express the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of a particle in cylindrical coordinates.

In rectangular coordinates the position vector is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the velocity and acceleration vectors are

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \quad \text{and} \quad \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}$$

In cylindrical coordinates, using Problem 4,

$$\begin{aligned}\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} &= (\rho\cos\phi)(\cos\phi\,\mathbf{e}_\rho - \sin\phi\,\mathbf{e}_\phi) \\ &\quad + (\rho\sin\phi)(\sin\phi\,\mathbf{e}_\rho + \cos\phi\,\mathbf{e}_\phi) + z\,\mathbf{e}_z \\ &= \rho\,\mathbf{e}_\rho + z\,\mathbf{e}_z\end{aligned}$$

Then $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\rho}{dt}\mathbf{e}_\rho + \rho\frac{d\mathbf{e}_\rho}{dt} + \frac{dz}{dt}\mathbf{e}_z = \dot{\rho}\,\mathbf{e}_\rho + \rho\,\dot{\phi}\,\mathbf{e}_\phi + \dot{z}\,\mathbf{e}_z$$

using Problem 5. Differentiating again,

$$\begin{aligned}\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} &= \frac{d}{dt}(\dot{\rho}\,\mathbf{e}_\rho + \rho\,\dot{\phi}\,\mathbf{e}_\phi + \dot{z}\,\mathbf{e}_z) \\ &= \dot{\rho}\frac{d\mathbf{e}_\rho}{dt} + \ddot{\rho}\,\mathbf{e}_\rho + \rho\,\dot{\phi}\frac{d\mathbf{e}_\phi}{dt} + \rho\,\ddot{\phi}\,\mathbf{e}_\phi + \dot{\rho}\,\dot{\phi}\,\mathbf{e}_\phi + \ddot{z}\,\mathbf{e}_z \\ &= \dot{\rho}\,\dot{\phi}\,\mathbf{e}_\phi + \ddot{\rho}\,\mathbf{e}_\rho + \rho\,\dot{\phi}(-\dot{\phi}\,\mathbf{e}_\rho) + \rho\,\ddot{\phi}\,\mathbf{e}_\phi + \dot{\rho}\,\dot{\phi}\,\mathbf{e}_\phi + \ddot{z}\,\mathbf{e}_z \\ &= (\ddot{\rho} - \rho\,\dot{\phi}^2)\,\mathbf{e}_\rho + (\rho\,\ddot{\phi} + 2\dot{\rho}\,\dot{\phi})\,\mathbf{e}_\phi + \ddot{z}\,\mathbf{e}_z\end{aligned}$$

Example

Evaluate $\iiint\limits_V (x^2+y^2+z^2)\,dx\,dy\,dz$ where V is a sphere having center at the origin and radius equal to a.

Do this in spherical coordinates.

$$x^2+y^2+z^2 = r^2$$

$$dx\,dy\,dz = dV = r^2 \sin\theta\, d\theta\, d\varphi$$

$$\therefore \int_V r^2 \cdot r^2 \sin\theta\, d\theta\, d\varphi\, dr = \int_0^a r^4\, dr \int_0^\pi \sin\theta\, d\theta \times \int_0^{2\pi} d\varphi$$

$$= 4\pi \cdot \frac{a^5}{5} = \frac{4\pi a^5}{5}.$$

What did we learn today:

Symmetry of a problem decides what coordinate to choose.

We introduced the idea of orthogonal curvilinear coordinates.

We realised that the scale factors were necessary to relate changes in arbitrary coordinate to a length scale.

We calculated the line, surface and volume elements in general in orthogonal curvilinear coordinates and specialised them to spherical and cylindrical polar coordinates.

Unlike Cartesian, the unit vectors are in general position dependent, which is a crucial point to note.