Physics II: Electromagnetism

PH 102

Lecture 5

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Quick Recap:

Orthogonal curvilinear coordinates $:(u_1, u_2, u_3)$, unit vectors $:(\hat{e}_1, \hat{e}_2, \hat{e}_3)$,

scale factors $:(h_1, h_2, h_3)$,

$$h_i \hat{e}_i = \frac{\partial \vec{r}}{\partial u_i} \qquad \hat{e}_i = \frac{\partial \vec{r}}{\partial u_i} \Big/ \left|\frac{\partial \vec{r}}{\partial u_i}\right| \qquad h_i = \left|\frac{\partial \vec{r}}{\partial u_i}\right|$$

Orthogonality of unit vectors: $\hat{e}_i . \hat{e}_j = \delta_{ij}$

$$d\vec{r} = h_1 \hat{e}_1 du_1 + h_2 \hat{e}_2 du_2 + h_3 \hat{e}_3 du_3$$

Gradient: $$\vec{\nabla} T(u_1, u_2, u_3) = \frac{1}{h_1} \frac{\partial T}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial T}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial T}{\partial u_3} \hat{e}_3$$

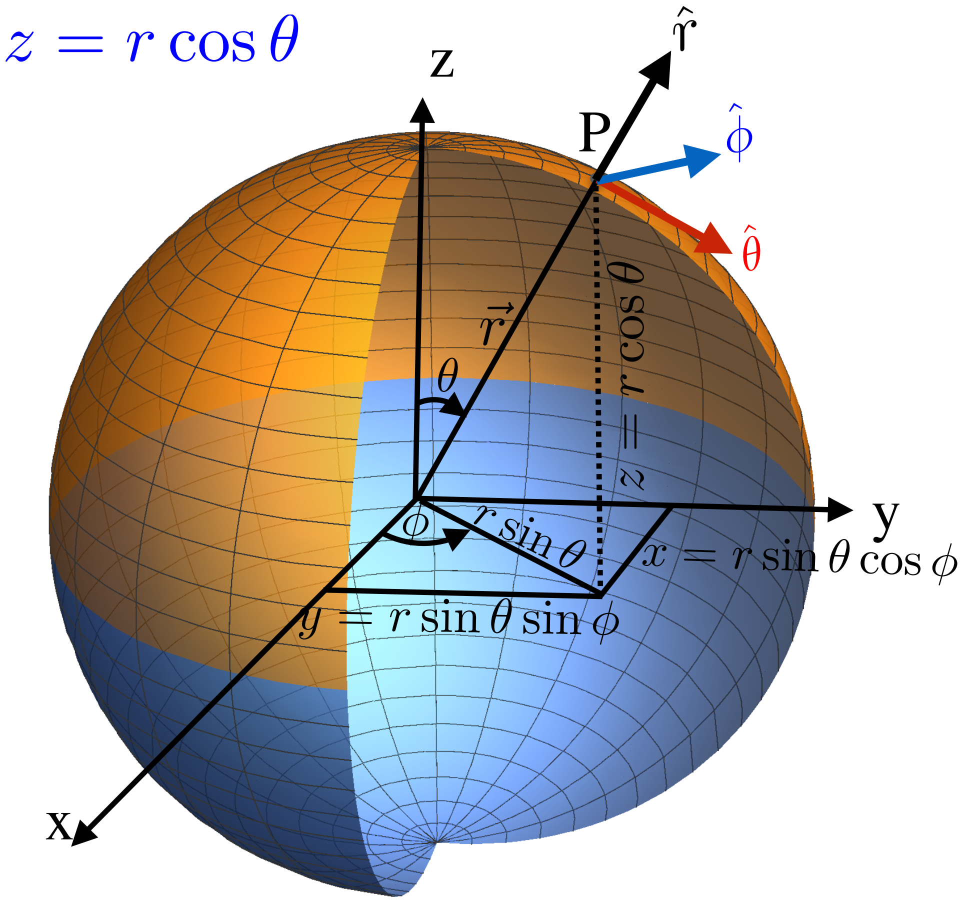
Divergence: $$\vec{\nabla} . \vec{V} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial (h_2 h_3 V_1)}{\partial u_1} + \frac{\partial (h_3 h_1 V_2)}{\partial u_2} + \frac{\partial (h_1 h_2 V_3)}{\partial u_3} \right)$$

Curl: $$\vec{\nabla} \times \vec{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

Laplacian: $$\nabla^2 \mathrm{T} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \mathrm{T}}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \mathrm{T}}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \mathrm{T}}{\partial u_3} \right) \right]$$

Spherical Polar Coordinates

- Range of r: $0 \leq r < \infty$ • Range of θ: $0 \leq \theta \leq \pi$ • Range of ϕ: $0 \leq \phi < 2\pi$
- Transformations: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$
- Unit vectors:



$$\hat{e}_1 \equiv \hat{r} = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$$

$$\hat{e}_2 \equiv \hat{\theta} = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$$

$$\hat{e}_3 \equiv \hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

Unit vectors in spherical polar coordinates are dependent on position

$$\frac{\partial \hat{r}}{\partial \theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} = \hat{\theta}$$

$$\frac{\partial \hat{r}}{\partial \phi} = \sin\theta(-\sin\phi\hat{x} + \cos\phi\hat{y}) = -\sin\theta\hat{\phi}$$

$$\frac{\partial \hat{\theta}}{\partial \theta} = -\sin\theta\cos\phi\hat{x} - \sin\theta\sin\phi\hat{y} - \cos\theta = -\hat{r}$$

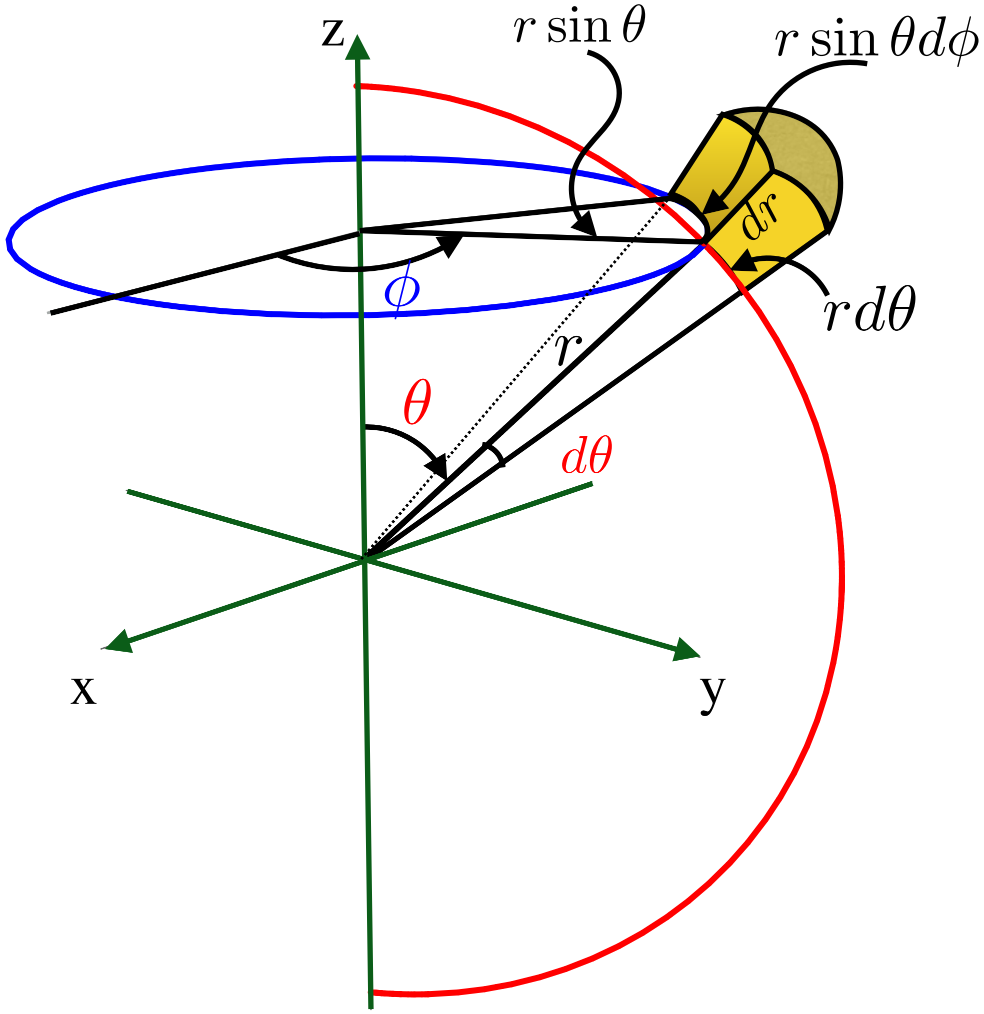
$$\frac{\partial \hat{\theta}}{\partial \phi} = \cos\theta(-\sin\phi\hat{x} + \cos\phi\hat{y}) = \cos\theta\hat{\phi}$$

$$\frac{\partial \hat{\phi}}{\partial \phi} = -\cos\phi\hat{x} - \sin\phi\hat{y} = -\sin\theta\hat{r} - \cos\theta\hat{\theta}$$

Spherical Polar Coordinates

Remember the space curves, where two coordinates were kept fixed.

Coordinate	Increment	Change in length scale along the coordinate	$h_i = \lvert \frac{\partial \vec{r}}{\partial u_i} \rvert$
r	dr	dr	1
θ	$d\theta$	$rd\theta$	r
ϕ	$d\phi$	$r\sin\theta d\phi$	$r\sin\theta$



Recall $d\tau = h_r h_\theta h_\phi dr d\theta d\phi = r^2 \sin\theta dr d\theta d\phi$.

The way to ``see" this:

The curved parallelepiped with length $r\sin\theta d\phi$, width $rd\theta$ and height dr has the volume $d\tau = (r\sin\theta d\phi)(rd\theta)dr = r^2\sin\theta dr d\theta d\phi$.

Spherical Polar Coordinates: Grad., Div., Curl, Laplacian:

Using the formulae for gradient, divergence, curl and Laplacian in orthogonal curvilinear coordinates, we can write them for the spherical polar:

Recall: $(u_1, u_2, u_3) \equiv (r, \theta, \phi)$ and $h_1 = h_r = 1, \;\; h_2 = h_\theta = r, \;\; h_3 = h_\phi = r \sin\theta$

$$\hat{e}_1 \equiv \hat{r}, \;\; \hat{e}_2 \equiv \hat{\theta}, \;\; \hat{e}_3 \equiv \hat{\phi}$$

Gradient:
$$\vec{\nabla} T(r, \theta, \phi) = \hat{r} \frac{\partial T}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial T}{\partial \theta} + \hat{\phi} \frac{1}{r \sin\theta} \frac{\partial T}{\partial \phi}$$

Divergence:
$$\vec{\nabla} . \vec{V}(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta V_\theta) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}(V_\phi)$$

Curl:
$$\vec{\nabla} \times \vec{V}(r, \theta, \phi) = \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta}(\sin\theta V_\phi) - \frac{\partial V_\theta}{\partial \phi} \right] \hat{r} + \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial V_r}{\partial \phi} - \frac{\partial}{\partial r}(r V_\phi) \right] \hat{\theta} + \frac{1}{r} \left[\frac{\partial}{\partial r}(r V_\theta) - \frac{\partial V_r}{\partial \theta} \right] \hat{\phi}$$

Laplacian:
$$\nabla^2 T(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \phi} \left(\sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 T}{\partial \phi^2}$$

H. W: Do the same analysis for Cylindrical Polar

Examples:

In Tutorial 1, we evaluated $\vec{\nabla} r^n$ using Cartesian coordinates and the calculation required quite a few steps. Finally, we arrived at the result $\vec{\nabla} r^n = n r^{n-1}\hat{r}$. The result can be arrived at in a single step if we take help of Spherical Polar Coordinates:

Using the form of the gradient operator in Spherical Polar Coordinate:

$$\vec{\nabla} \equiv \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\vec{\nabla} r^n = \hat{r}\frac{\partial}{\partial r} r^n = \hat{r}\ n r^{n-1} \qquad \text{(Since } r \text{ has no dependence on } \theta, \phi.)$$

In a similar manner, you can show the following:

$$\begin{aligned}
\vec{\nabla}.(\hat{r} f(r)) &= \frac{2}{r} f(r) + \frac{df}{dr} \\
\vec{\nabla}.(\hat{r} r^n) &= (n+2) r^{n-1} \\
\vec{\nabla} \times (\hat{r} f(r)) &= 0 \\
\nabla^2 f(r) &= \frac{2}{r}\frac{df}{dr} + \frac{d^2 f}{dr^2} \\
\nabla^2 r^n &= n(n+1) r^{n-2}.
\end{aligned}$$

Examples:

Remember

$$\vec{\nabla}\times\vec{V} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1V_1 & h_2V_2 & h_3V_3 \end{vmatrix}$$

Calculate $\vec{\nabla}\times(\vec{\nabla}\times\hat{\phi}A_\phi(r,\theta))$.

It is evident that the coordinate used is spherical polar coordinates. The ϕ component of the vector $\vec{A}$ is a function of r,θ.

$$\begin{aligned}\vec{\nabla}\times(\vec{\nabla}\times\hat{\phi}A_\phi(r,\theta)) &= \vec{\nabla}\times\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta A_\phi(r,\theta)\end{vmatrix} \\ &= \vec{\nabla}\times\frac{1}{r^2\sin\theta}\left[\hat{r}\frac{\partial}{\partial\theta}(r\sin\theta A_\phi) - r\hat{\theta}\frac{\partial}{\partial r}(r\sin\theta A_\phi)\right]\end{aligned}$$

Taking the curl a second time:

$$\vec{\nabla}\times(\vec{\nabla}\times\hat{\phi}A_\phi(r,\theta)) = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta A_\phi) & -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(r\sin\theta A_\phi) & 0\end{vmatrix}$$

$$= -\hat{\phi}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_\phi) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right)\right]$$

You may encounter this type of calculations when magnetic vector potentials will be discussed

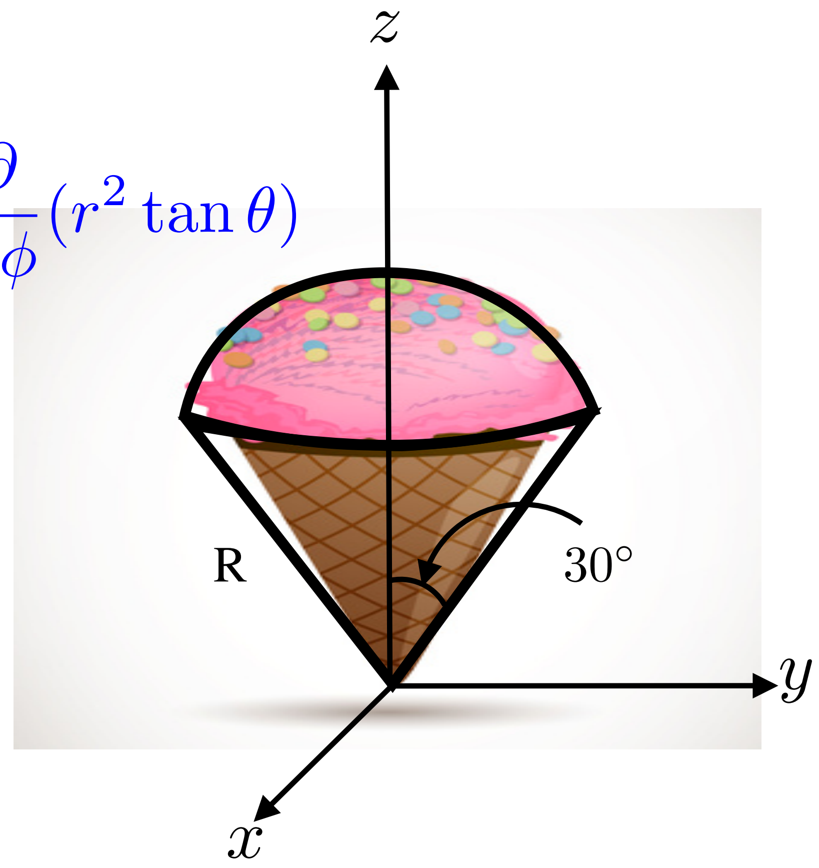
More examples:

Check the divergence theorem for the function $\vec{A} = r^2 \sin\theta \hat{r} + 4r^2 \cos\theta \hat{\theta} + r^2 \tan\theta \, \hat{\phi}$ using the volume of the "ice-cream" cone.

Divergence of $\vec{A}$:

$$\vec{\nabla}.\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2 \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta 4r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\tan\theta)$$

$$= \frac{1}{r^2}4r^3\sin\theta + \frac{1}{r\sin\theta}4r^2(\cos^2\theta - \sin^2\theta) = 4r\frac{\cos^2\theta}{\sin\theta}$$



Therefore,

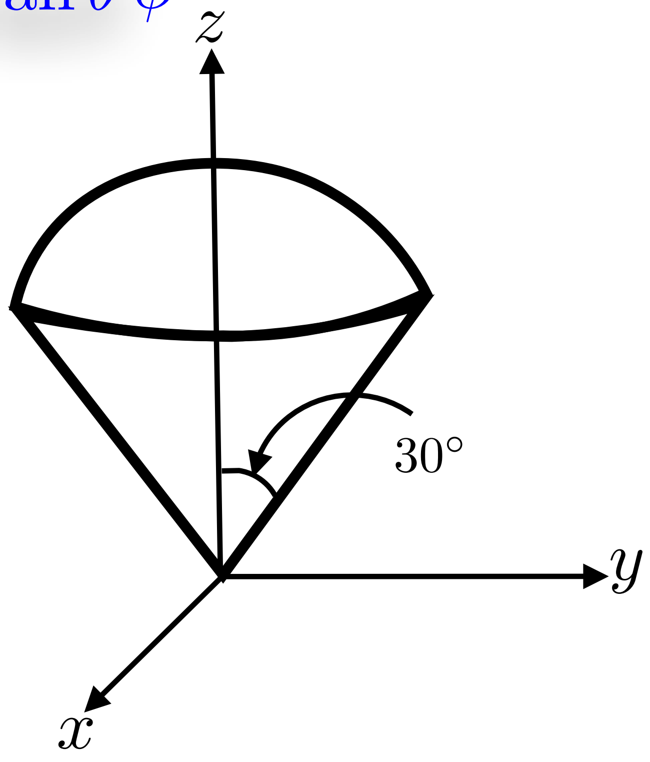
$$\int(\vec{\nabla}.\vec{A})d\tau = \int\left(4r\frac{\cos^2\theta}{\sin\theta}\right)(r^2\sin\theta dr d\theta d\phi)$$

$$= \int_{r=0}^{R} 4r^3 dr \int_{\theta=0}^{\pi/6}\cos^2\theta d\theta \int_{\phi=0}^{2\pi} d\phi \quad = (R^4)(2\pi)\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]\Bigg|_0^{\pi/6}$$

$$= 2\pi R^4\left(\frac{\pi}{12} + \frac{\sin 60^\circ}{4}\right) = \frac{\pi R^4}{12}(2\pi + 3\sqrt{3})$$

Let us check this result by directly calculating the surface integral

More examples (contd.): $\vec{A} = r^2 \sin\theta \hat{r} + 4r^2 \cos\theta \hat{\theta} + r^2 \tan\theta \,\hat{\phi}$



The surface consists of two parts:

(i) The "ice cream": For which $r = R$; $\phi : 0 \to 2\pi$; $\theta : 0 \to \pi/6$ and $d\vec{a} = R^2 \sin\theta d\theta d\phi \hat{r}$.

Therefore $\vec{A}.d\vec{a} = (R^2 \sin\theta)(R^2 \sin\theta d\theta d\phi) = R^4 \sin^2\theta d\theta d\phi$.

$$\int \vec{A}.d\vec{a} = R^4 \int_0^{\pi/6} \sin^2\theta d\theta \int_0^{2\pi} d\phi = \frac{\pi R^4}{6}\left(\pi - 3\frac{\sqrt{3}}{2}\right)$$

(ii) The "cone":

For which $\theta = \pi/6$; $\phi : 0 \to 2\pi$; $r : 0 \to R$ and $d\vec{a} = r\sin\theta d\phi dr \hat{\theta} = \frac{1}{2} r d\phi dr \hat{\theta}$

Therefore $\vec{A}.d\vec{a} = \left(\frac{1}{2} r d\phi dr\right)(4r^2 \cos\theta) = \sqrt{3} r^3 d\phi dr$, (since $\cos(\pi/6) = \sqrt{3}/2$)

$$\int \vec{A}.d\vec{a} = \sqrt{3}\int_0^R r^3 dr \int_0^{2\pi} d\phi = \sqrt{3}.\frac{R^4}{4}.2\pi = \frac{\sqrt{3}}{2}\pi R^4$$

$\therefore$ Total contribution $\int \vec{A}.d\vec{a} = \frac{\pi R^4}{6}\left(\pi - 3\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\pi R^4$

$$= \frac{\pi R^4}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \sqrt{3}\right) = \frac{\pi R^4}{12}(2\pi + 3\sqrt{3})$$

Example

Find the volume of the smaller of the two regions bounded by the sphere $x^2 + y^2 + z^2 = 16$ and the cone $z^2 = x^2 + y^2$.

Use cylindrical coordinates.

Range of coordinates : ϕ : $\phi = 0 \rightarrow 2\pi$

z : $z^2 + s^2 = 16 \Rightarrow z = \sqrt{16 - s^2}$

to

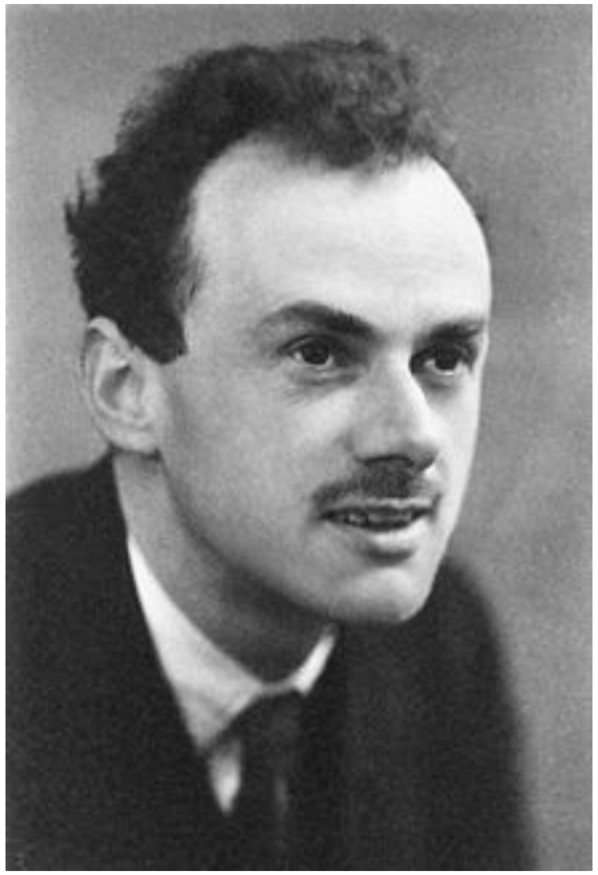
$z^2 = s^2 \Rightarrow z = s$

s : $s = 0$ to $2s^2 = 16 \Rightarrow s = 2\sqrt{2}$

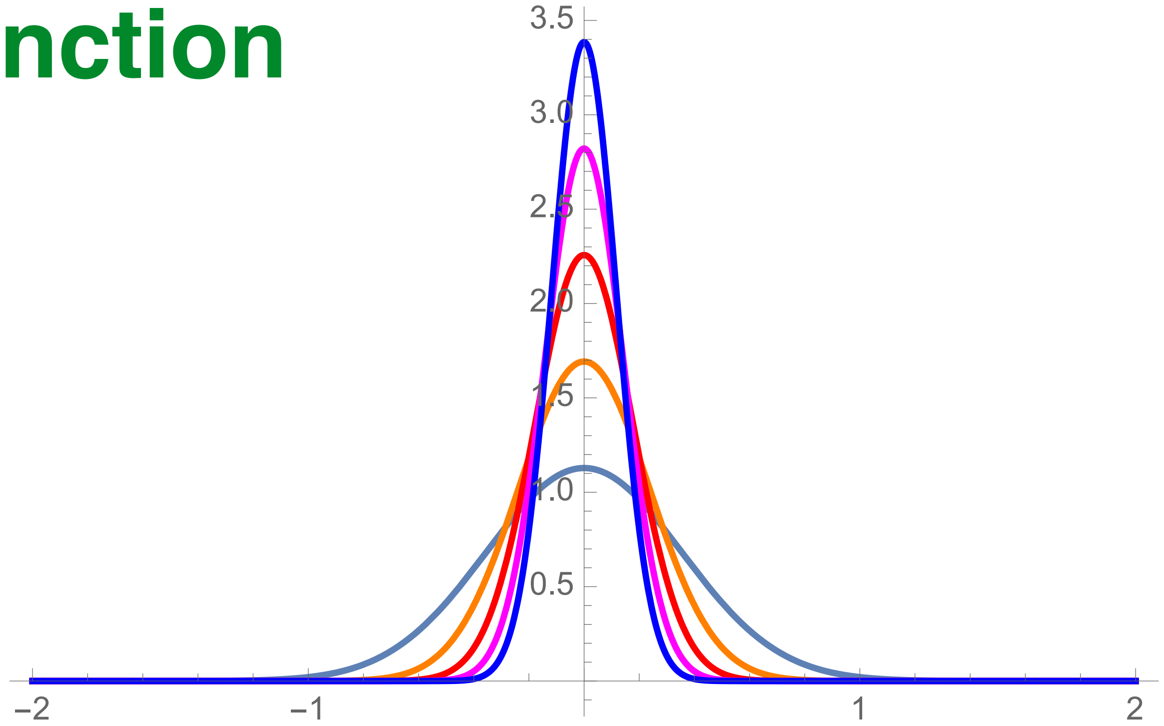
$$\therefore \iiint_V s \, ds \, d\phi \, dz$$

$$= \int_{\phi=0}^{2\pi} d\phi \int_{s=0}^{2\sqrt{2}} s \, ds \int_{\sqrt{16-s^2}}^{s} dz = \frac{64\pi(\sqrt{2} - 2)}{3}$$

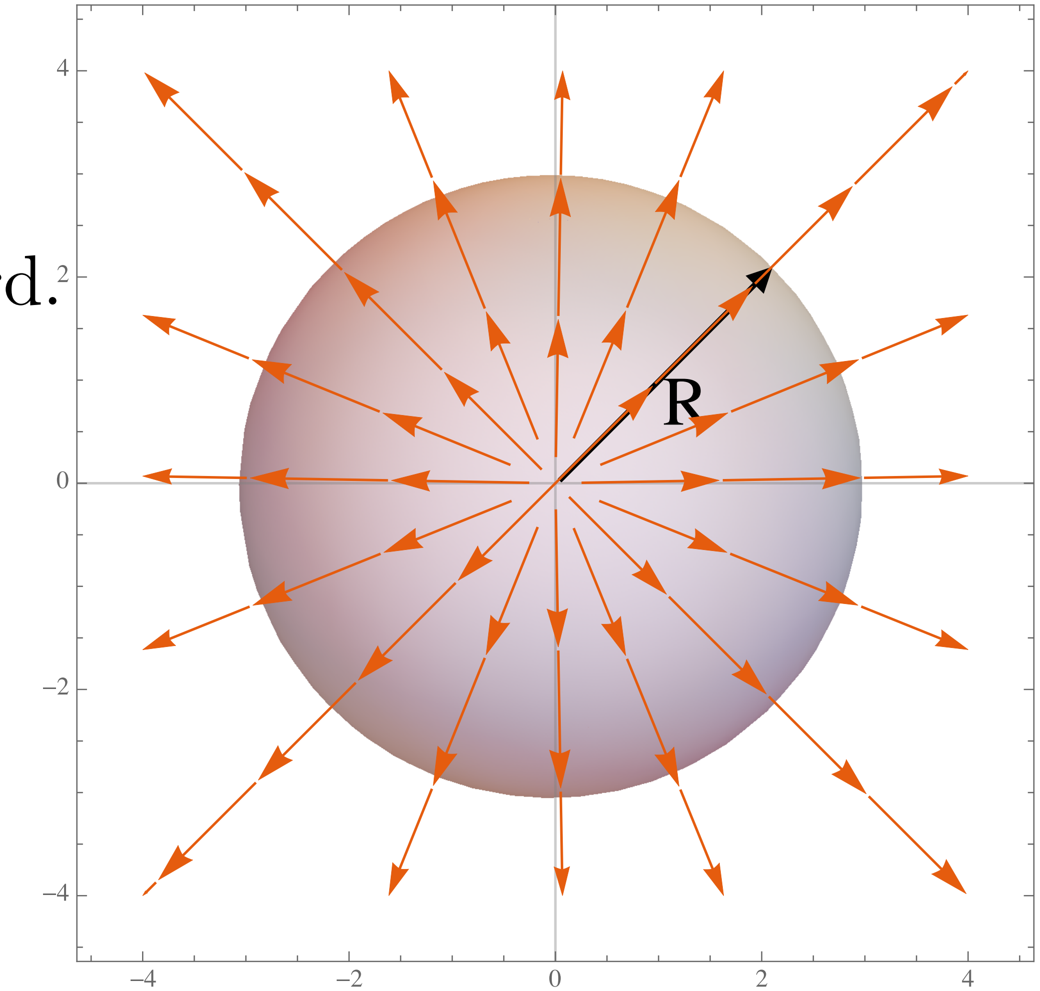
$$\therefore \text{Volume} = \left| \iiint_V dv \right| = \frac{64\pi}{3}(2 - \sqrt{2})$$



Dirac Delta Function



Divergence of $\vec{V} = \frac{\hat{r}}{r^2}$



At every direction, $\vec{V}$ is directed radially outward.

The function has large positive divergence.

But…

$$\vec{\nabla}.\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

$$\Longrightarrow \int_{\mathcal{V}} (\vec{\nabla}.\vec{V})\, d\tau = 0$$

However, more problem arises if you try to apply divergence theorem to $\vec{V}$

Suppose, we integrate over a sphere of radius R, entered at origin: the surface integral is

$$\oint \vec{V}.d\vec{a} = \int \left(\frac{1}{R^2}\hat{r}\right).(R^2 \sin\theta d\theta d\phi \hat{r})$$

$$= \left(\int_0^{\pi} \sin\theta d\theta\right)\left(\int_0^{2\pi} d\phi\right) = 4\pi$$

But divergence theorem states that $\int_{\mathcal{V}}(\vec{\nabla}.\vec{V})d\tau = \int_{\mathcal{S}} \vec{V}.d\vec{a}$!

What is happening here? Is divergence theorem wrong??

Divergence of $\vec{V} = \frac{\hat{r}}{r^2}$

The source of the problem is the point r=0, where the function blows up!

It is true that $\vec{\nabla}.\vec{V} = 0$ everywhere except at the origin. But, right at the origin the situation is more complicated.

Note that surface integral is independent of R; so if divergence theorem is right (and it is), we should expect $\int(\vec{\nabla}.\vec{V})d\tau = 4\pi$. The entire contribution must then be coming from the point $r = 0$.

$\vec{\nabla}.\vec{V}$ has the bizarre property that it vanishes everywhere except at one point, and yet its integral over any volume containing that point is $4\pi \implies$ "No Ordinary Function".

Dirac Delta Function

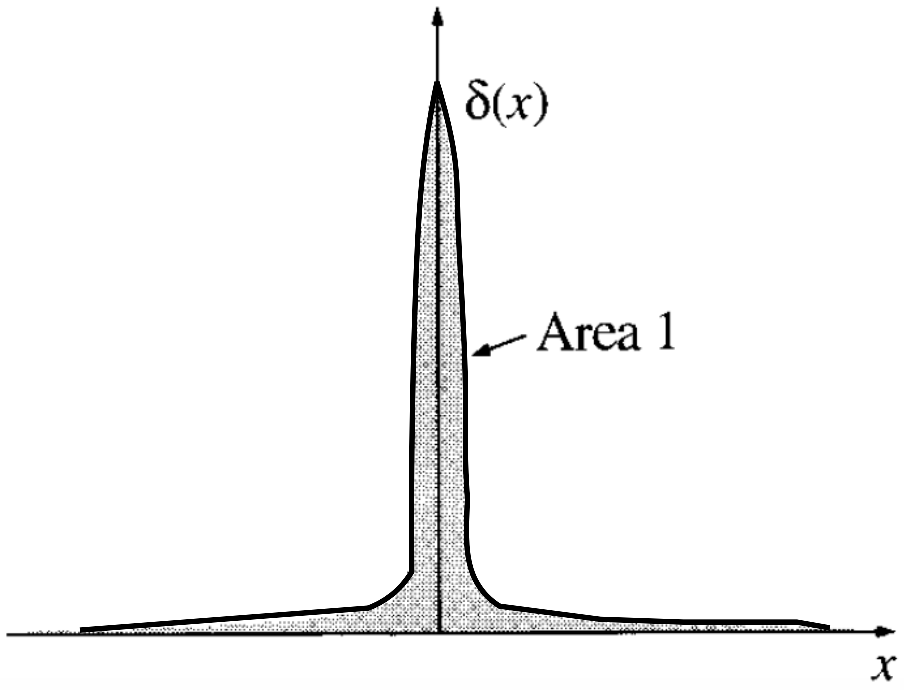
Dirac Delta Function

A real function δ on $\mathbb{R}$ is called Dirac Delta Function

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0; \\ \infty & \text{if } x = 0. \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x)dx = 1.$$



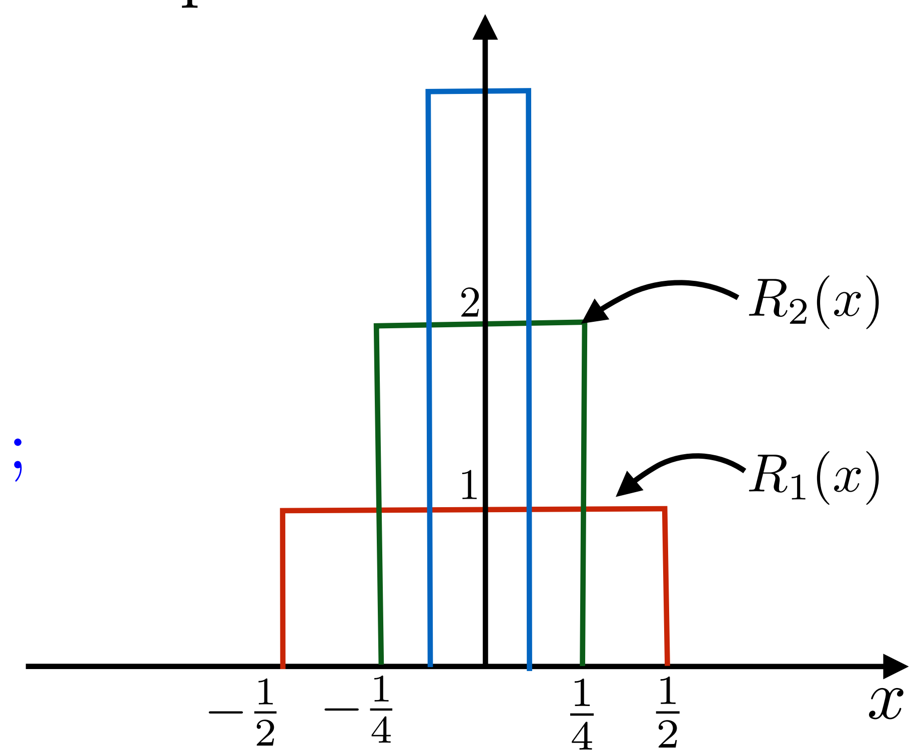
"Infinitely high, infinitesimally narrow spike with area 1"

This of course is a heuristic definition. Not well defined at x=0

In a strict sense, it is not a function and mathematicians would like to call it as "generalised function" or a "distribution".

Then, how to "see" them?

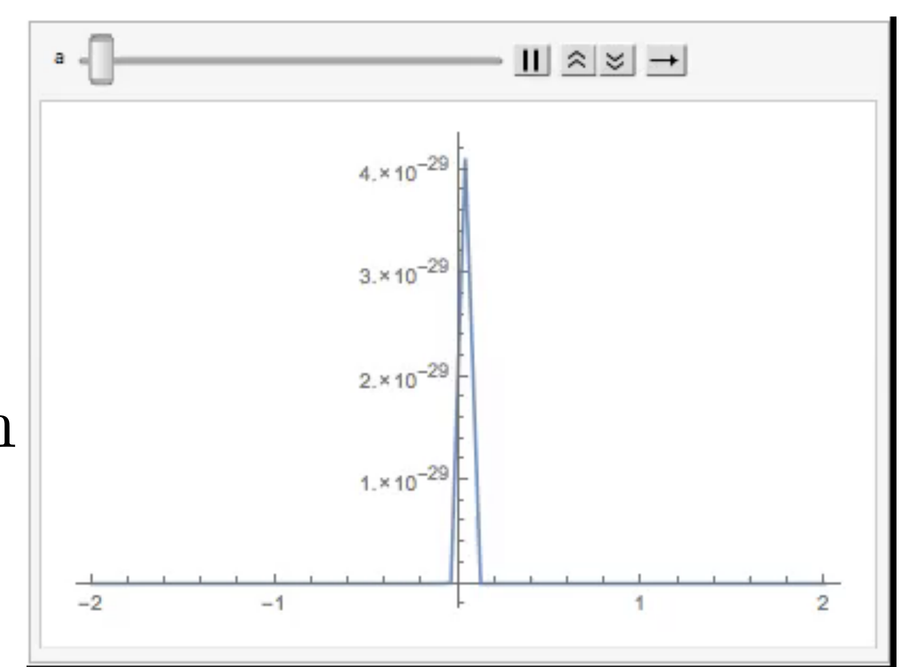
The best way to look at a delta function is as a limit of a sequence of functions. We give a few such examples:



★ We can have a sequence of function as

$$R_n(x) = \begin{cases} 0 & \text{if } x \leq -\frac{1}{2^n}; \\ 2^{n-1} & \text{if } -\frac{1}{2^n} < x < \frac{1}{2^n}; \\ 0 & \text{if } x > \frac{1}{2^n}. \end{cases}$$

For a fixed n, it represents a rectangle of height n and width between $-\frac{1}{2^n}$ to $\frac{1}{2^n}$. As $n \to \infty$, width decreases but height increases in such a proportion that the area always remains 1. So, as $n \to \infty$, $R_n \to \delta$.



★ Consider the function $\delta_a(x) = \frac{1}{\sqrt{2\pi}a} e^{-x^2/2a^2}$ defined in such a way that $\int_{-\infty}^{\infty} \delta_a(x) dx = 1$ for any a. Then in the limit $a \to 0$, $\delta_a(x) \to \delta(x)$.

Dirac Delta Function: Properties

⋆ For a continuous function $f(x)$,

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

This means that for a continuous function $f(x)$, the product $f(x)\delta(x)$ is zero everywhere except at $x = 0$. It follows: $f(x)\delta(x) = f(0)\delta(x)$.

⋆ Translation: $\delta(x-a) = \left\{ \begin{array}{ll} 0 & \text{if } x \neq a \\ \infty & \text{if } x = a \end{array} \right\}$ with $\int_{\infty}^{\infty} \delta(x-a)dx = 1$

Therefore the first property tells us $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$

⋆ Although δ itself is not a legitimate function, integrals over δ are perfectly acceptable. In fact two expressions involving delta functions (say, $D_1(x)$ and $D_2(x)$) are called equal if $\int_{-\infty}^{\infty} f(x)D_1(x)dx = \int_{-\infty}^{\infty} f(x)D_2(x)dx$, for all $f(x)$.

⋆ Scaling : $\delta(kx) = \frac{1}{|k|}\delta(x)$, where k is any constant.

Infact, this property tells us $\delta(-x) = \delta(x)$.

Dirac Delta Function: Properties

Scaling : $\delta(kx) = \frac{1}{|k|}\delta(x)$, where k is any constant.

Proof: Chose an arbitrary test function $f(x)$ and consider the integral:

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx$$

Let $y \equiv kx$, so that $x = y/k$ and $dx = dy/k$. If $k > 0$, the integration limits are unchanged but if $k < 0$, the $x = \infty$ implies $y = -\infty$, and vice versa. Restoring the proper order of the limits:

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \pm\int_{-\infty}^{\infty} f(y/k)\delta(y)\frac{dy}{k} = \pm\frac{1}{k}f(0) = \frac{1}{|k|}f(0)$$

Therefore, under the integral sign, $\delta(kx)$ serves the same purpose as $(1/|k|)\delta(x)$:

$$\int_{-\infty}^{\infty} f(x)\delta(kx) = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{|k|}\delta(x)\right].$$

Dirac Delta Function: in three dimensions

Generalize in 3-D: $\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$

This 3-D Dirac Delta is zero everywhere except at origin (0,0,0), with its volume integral being 1

$$\int_{\text{all space}} \delta^3(\vec{r})d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dxdydz = 1$$

Generalizing $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(z)$ in 3-D: $\int_{\text{all space}} f(\vec{r})\delta^3(\vec{r}-\vec{r_0})d\tau = f(\vec{r_0})$

Let us get back to the divergence paradox :

Recall that $\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) = 0$, if $\vec{r} \neq 0$.

The one and only point where divergence is non-zero is origin.

But do we know the value of the divergence at origin? NO!

Assume that it is $k\delta^3(\vec{r})$

Divergence theorem $\Longrightarrow \int_{\mathcal{V}} \left(\vec{\nabla}.\frac{\hat{r}}{r^2}\right) d\tau = \oint_{\mathcal{S}} \frac{\hat{r}}{r^2}.d\vec{a} \Longrightarrow k\int_{\mathcal{V}} \delta^3(\vec{r})d\tau = 4\pi \Longrightarrow k = 4\pi$

($\oint_{\mathcal{S}} \frac{\hat{r}}{r^2}.d\vec{a}$ → 4π)

$$\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$$

Few examples:

1. Evaluate $\int_0^3 x^3 \delta(x-2) dx$.

   The delta function picks out the value of x^3 at the point $x = 2$, so the integral is $2^3 = 8$. Note however, if the upper limit had been 1 (instead of being 3), the answer would be 0, because the spike would then be outside the domain of integration.

2. Evaluate $\int_2^6 (3x^2 - 2x - 1)\delta(x-3) dx$.

   Recall that $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$. Here $f(x) = (3x^2 - 2x - 1)$, $a = 3$ and it lies between the limits of the integration. Therefore $\int_2^6 (3x^2 - 2x - 1)\delta(x-3)dx = f(3) = 20$.

3. Evaluate $\int_{-2}^{2} (2x+3)\delta(3x)dx$.

   Change variable $x = t/3$. Then $\displaystyle\int_{-2}^{2} (2x+3)\delta(3x)dx = \int_{-\frac{2}{3}}^{\frac{2}{3}} \left(2\frac{t}{3} + 3\right) \delta(t)\frac{dt}{3} = 1$

   Alternatively, you can use $\delta(3x) = \delta(x)/3$ and proceed accordingly.

4. Evaluate $J = \int_{\mathcal{V}} (r^2 + 2)\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) d\tau$. Here $\mathcal{V}$ is a sphere of radius R centred at origin.

$$J = \int_{\mathcal{V}} (r^2 + 2) 4\pi \delta^3(\vec{r}) d\tau = 4\pi(0 + 2) = 8\pi$$