Physics II: Electromagnetism

PH 102

Lecture 7

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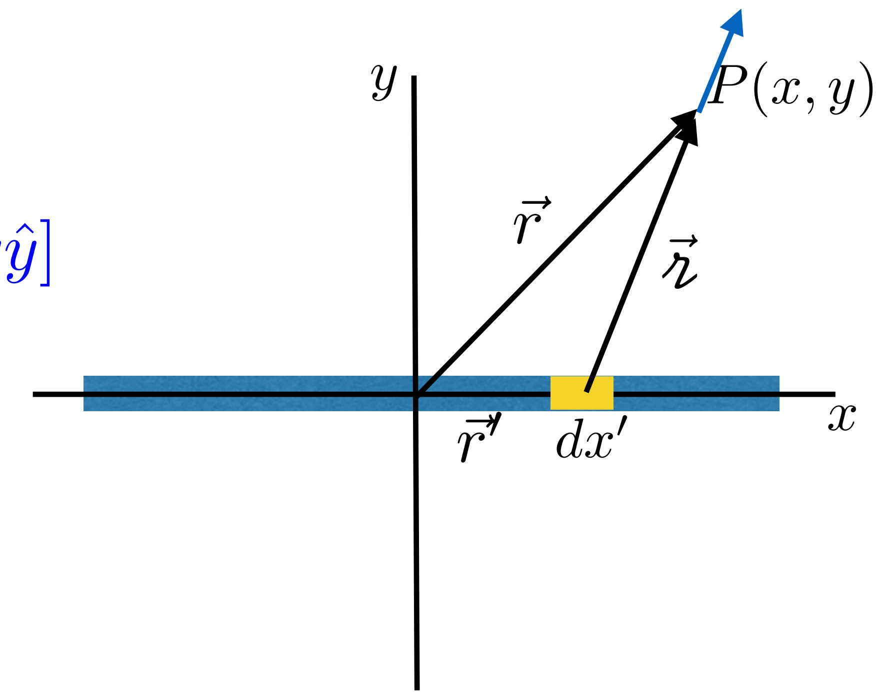
Recap: Electric field for infinite line charge

$$\vec{\mathscr{r}} = (x - x')\hat{x} + y\hat{y}$$

The field at $P(x,y)$ due to element dx' at $(x',0)$ is:

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda dx'}{\mathscr{r}^3}\vec{\mathscr{r}} = \frac{1}{4\pi\epsilon_0}\frac{\lambda dx'}{[(x-x')^2+y^2]^{3/2}}[(x-x')\hat{x}+y\hat{y}]$$



$$E_x = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{(x-x')dx'}{[(x-x')^2+y^2]^{3/2}} = -\frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{zdz}{(z^2+y^2)^{3/2}}$$

$$= 0$$

(where $z = x - x'$)

$$E_y = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{ydx'}{[(x-x')^2+y^2]^{3/2}} = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{ydz}{(z^2+y^2)^{3/2}}$$

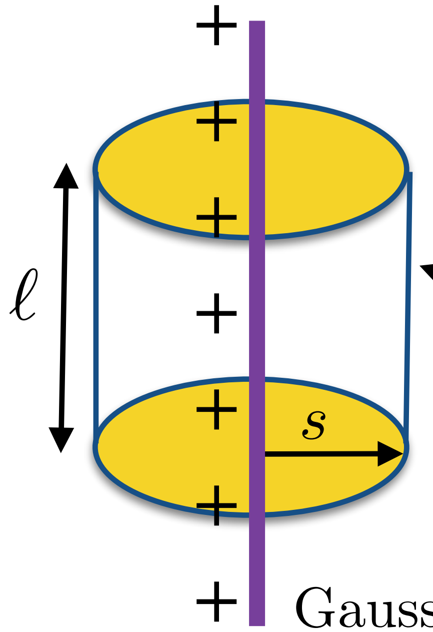
Substitute $z = y\tan\theta$:

$$= \frac{\lambda}{4\pi\epsilon_0}\int_{-\pi/2}^{\pi/2}\frac{y^2\sec^2\theta}{y^3\sec^3\theta}d\theta = \frac{\lambda}{4\pi\epsilon_0}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta \quad \boxed{= \frac{\lambda}{2\pi\epsilon_0 y}}$$

Field of an infinite line charge placed on the x-axis: $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 y}\hat{y}$

Gauss's Law: More examples: Infinite line charge

Field for a line charge:



- Line charge density λ

- If you like you could consider a solid cylinder with uniform charge density and then send radius to zero.

- want to know the electric field due to this line of charge

- Our set-up has cylindrical symmetry.

- Choose Gaussian surface to be a cylinder of height ℓ and width s

$$\oint_{\mathcal{S}} \vec{E}.d\vec{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \lambda \ell$$

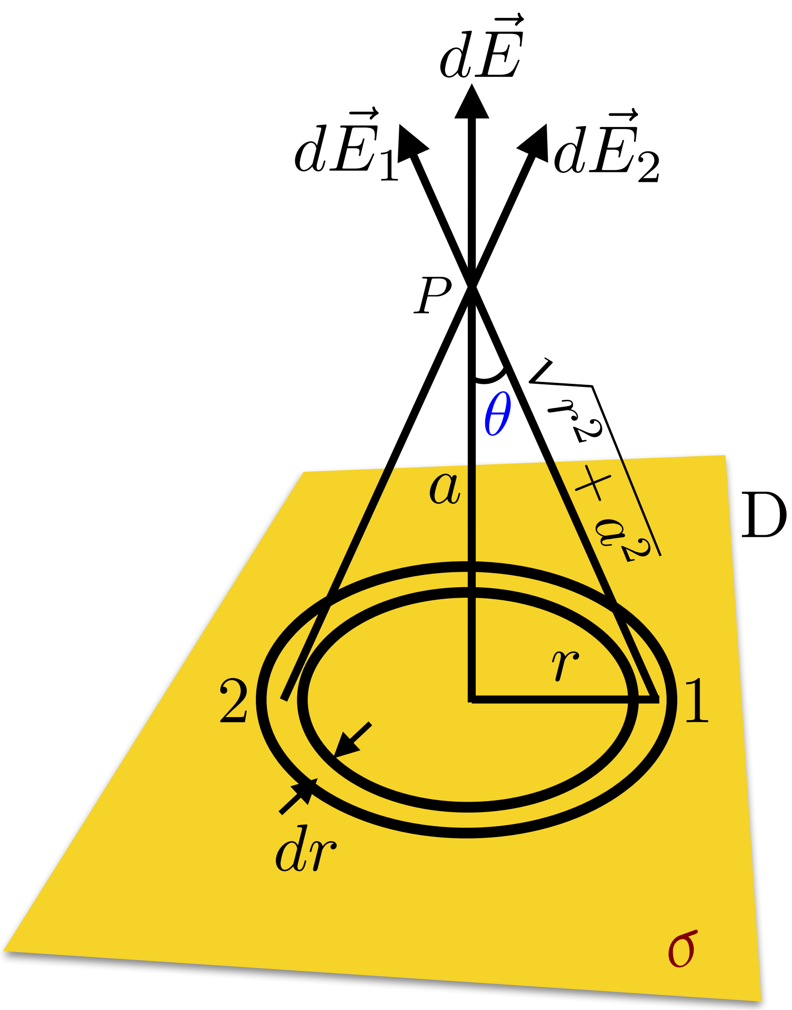
- By symmetry the electric field points in the radial direction, away from the line. Denote this vector in cylindrical polar coordinates $\vec{E} = E(s)\hat{s}$

- Two end caps of Gaussian surface don't contribute to the integral because their normal points in the $\hat{z}$ direction and $\hat{z}.\hat{s} = 0$.

$$\therefore \quad \oint_{\mathcal{S}} \vec{E}.d\vec{a} = E(s) 2\pi s \ell \quad = \frac{1}{\epsilon_0} \lambda \ell \quad \Longrightarrow \quad \vec{E}(s) = \frac{\lambda}{2\pi\epsilon_0 s} \hat{s}$$

Field due to a line charge drops off more slowly (~1/r) than a point charge (~1/r²)

Electric field for infinite surface charge



- Symmetry: Electric field at a height a should be independent of two coordinates parallel to plane.
- Symmetry: Electric field should always be directed perpendicular to the plane.

Draw $\perp$ to the plane passing through P on which we want the field.

Divide ring into concentric rings of radius r and width dr.

Find contribution $d\vec{E}$ from each ring and integrate over all rings.

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\,2\pi r\sigma dr\,\frac{1}{r^2+a^2}\cos\theta = \frac{2\pi r\sigma dr}{4\pi\epsilon_0}\frac{1}{r^2+a^2}\frac{a}{\sqrt{r^2+a^2}}$$

Set $r = au$, i.e. $dr = adu$, $\quad E = \dfrac{2\pi\sigma}{4\pi\epsilon_0}\displaystyle\int_0^\infty \frac{au\,adu}{a^2(1+u^2)}\frac{a}{a\sqrt{1+u^2}} = \frac{\sigma}{2\epsilon_0}\int_0^\infty \frac{udu}{(1+u^2)^{3/2}}$

The integral is 1 as can be seen by choosing $1+u^2 = t$, $\quad = \dfrac{\sigma}{2\epsilon_0}$

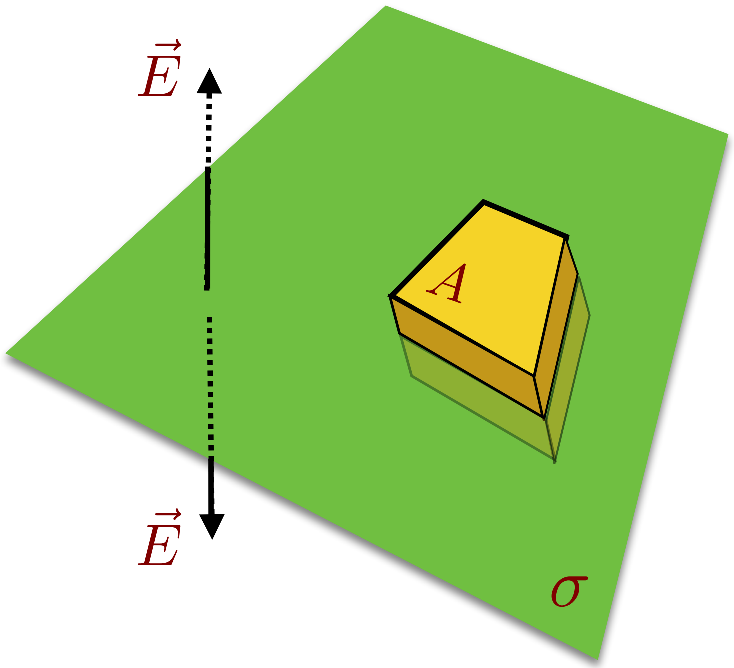
Field does not decrease as we move away from the plane. It is independent of “a”

Why? $\quad$ Dimensional argument. $\quad$ Geometric argument.

Gauss's Law: More examples: Infinite plane with surface charge

Field for a surface charge:

- Surface charge density σ
- Choose Gaussian surface to be a Gaussian "pillbox"
- Extending equal distances above and below the plane.
- It may also be cylindrical shaped
- Symmetry $\implies$ field is perpendicularly away from plane with the same magnitude at each point above and below.



$$\oint_{\mathcal{S}} \vec{E}.d\vec{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \sigma A \quad (A: \text{area of the lid of "pillbox"})$$

- $\therefore$ The top and bottom surfaces yield $\int \vec{E}.d\vec{a} = 2A|\vec{E}|$
- side surfaces contribute nothing. Hence $\boxed{\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n}}$

  where $\hat{n}$ is unit vector pointing away from the surface.

The result is independent of how far away you are! The more you move farther away from the plane, more and more charge comes to your field of view and this compensates the diminishing influence of any particular piece.

Curl of the Electric Field

Suppose a point charge q is placed at the origin. Then the electric field at a point P which is at a distance $\vec{r}$ is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\vec{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}(x\hat{x} + y\hat{y} + z\hat{z})$$

The curl of the electromagnetic field is

Using Cartesian

$$\vec{\nabla} \times \vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x/r^3) & (y/r^3) & (z/r^3) \end{vmatrix}$$

$$[\vec{\nabla} \times \vec{E}(\vec{r})]_x = \frac{q}{4\pi\epsilon_0}\left[\frac{\partial}{\partial y}(z/r^3) - \frac{\partial}{\partial z}(y/r^3)\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\left(-\frac{3yz}{r^5}\right) - \left(-\frac{3yz}{r^5}\right)\right] = 0$$

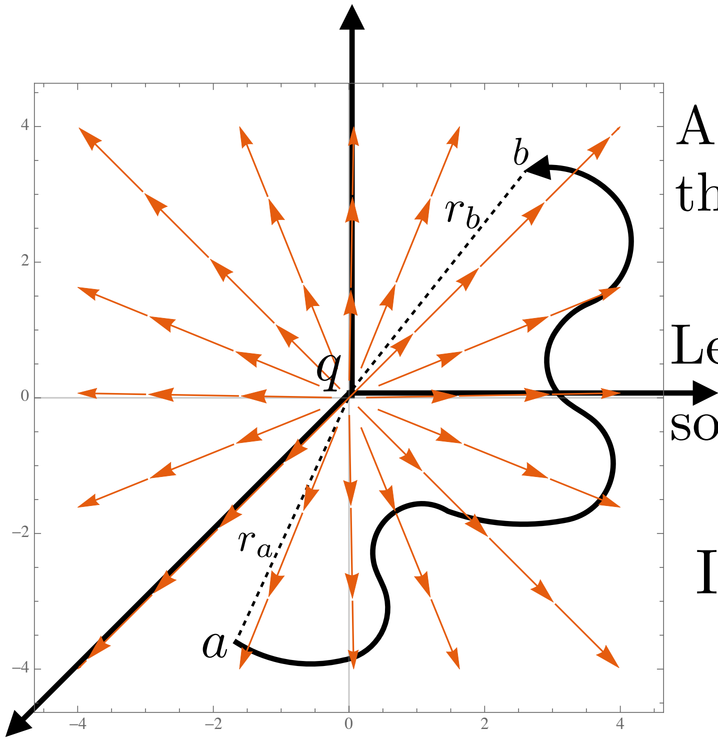
Using Spherical Polar

$$\vec{\nabla} \times \vec{E}(\vec{r}) = \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 1/r^2 & 0 & 0 \end{vmatrix}$$

$$= 0$$

Similarly the other components = 0

Curl of the Electric Field



A look at the electric field lines for a point charge will tell you that the curl of the electric field is zero!

Let us be a bit more rigorous and calculate the line integral of this field from some point a to some other point b $\int_a^b \vec{E}.d\vec{\ell}$

In spherical coordinates $d\vec{\ell} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$, so that

$$\vec{E}.d\vec{\ell} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$$

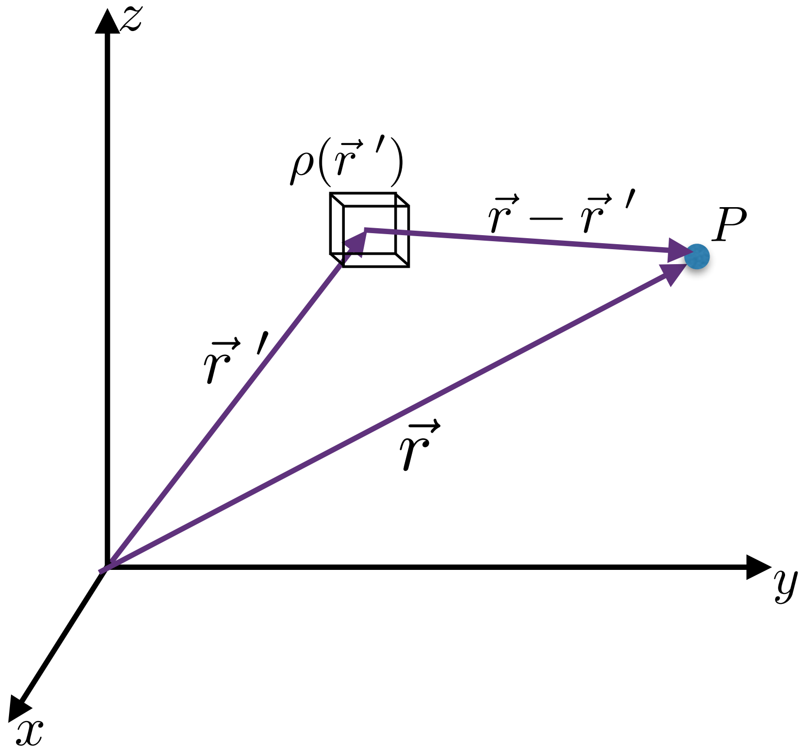
$$\therefore \int_a^b \vec{E}.d\vec{\ell} \quad = \quad \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}dr$$

$$= \quad -\frac{1}{4\pi\epsilon_0}\frac{q}{r}\Big|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_b} - \frac{q}{r_a}\right)$$

This integral around a closed path is evidently zero, since $r_a = r_b$ for a closed path $\implies \oint \vec{E}.d\vec{\ell} = 0$

Applying Stoke's theorem $\vec{\nabla} \times \vec{E} = 0$

Curl of the Electric Field (General case)



Extending the result to arbitrary charge distribution $\rho(\vec{r}\,')$:

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}d\tau'}$$

We need to take the curl with respect to the variable $\vec{r}$:

$$\begin{aligned}\vec{\nabla}\times\vec{E}(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\vec{\nabla}\times\int \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}d\tau' \\ &= \frac{1}{4\pi\epsilon_0}\int \rho(\vec{r}\,')\left(\vec{\nabla}\times\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}d\tau'\right) = 0\end{aligned}$$

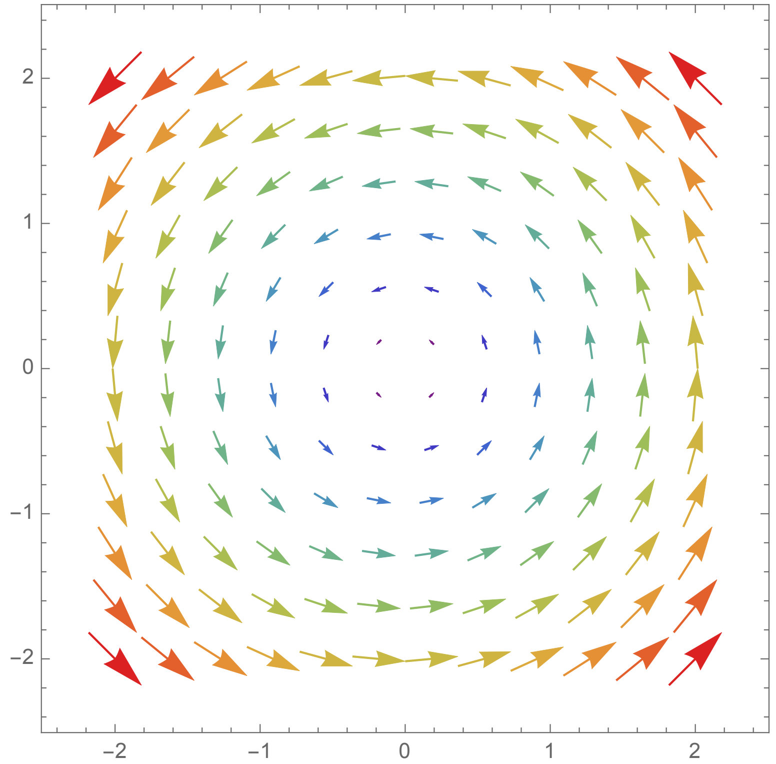
Recall

$$\left(\vec{\nabla}\times\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right)_x = -3(z-z')\frac{(y-y')}{|\vec{r}-\vec{r}\,'|^3} + 3(y-y')\frac{(z-z')}{|\vec{r}-\vec{r}\,'|^3}) = 0$$

Curl of an electric field is always zero

Electric Potential

Electric field $\vec{E}$ is not just <u>any</u> vector field: it is a special type of field whose curl is zero.



Ex: $\vec{E} = -y\hat{x} + x\hat{y}$ can not be an electric field since its curl is given by $2\hat{z}$.

We are going to use this special property of $\vec{E}$ to reduce a *vector problem* (finding $\vec{E}$) to a much simpler *scalar problem.*

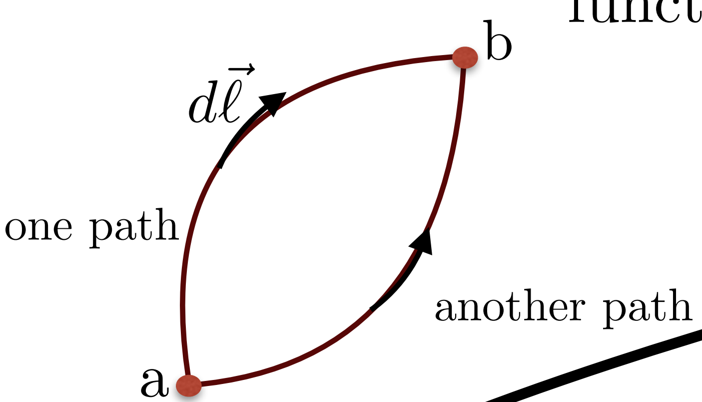
Any vector whose curl is zero is equal to the gradient of some scalar!

$$\vec{\nabla} \times \vec{E} = 0 \implies \vec{E} = -\vec{\nabla} V$$

Electric Potential

$\vec{\nabla}\times\vec{E}=0 \implies$ the line integral of $\vec{E}$ around any closed loop is zero by Stoke's theorem. Because $\oint \vec{E}.d\vec{\ell}=0$, the line integral of $\vec{E}$ from a to b is the same for all path.

Because of the fact the line integral is independent of the path, we define a function



$$V(r) = -\int_{\mathcal{O}}^{r} \vec{E}(\vec{r}\,').d\vec{\ell}'$$

electric potential

standard reference point

Potential difference between a and b

$$V(b)-V(a) = -\int_{\mathcal{O}}^{b} \vec{E}.d\vec{\ell} + \int_{\mathcal{O}}^{a} \vec{E}.d\vec{\ell} = -\int_{\mathcal{O}}^{b} \vec{E}.d\vec{\ell} - \int_{a}^{\mathcal{O}} \vec{E}.d\vec{\ell} = -\int_{a}^{b} \vec{E}.d\vec{\ell}$$

Using fundamental theorem for gradients : $V(b)-V(a)=\int_a^b(\vec{\nabla}V).d\vec{\ell}$, we get

$$\int_a^b (\vec{\nabla}V).d\vec{\ell} = -\int_a^b \vec{E}.d\vec{\ell}.$$

Potential difference is independent of the reference point $\mathcal{O}$

differential form of $V(r)=-\int_{\mathcal{O}}^{r}\vec{E}.d\vec{\ell}$

$$\vec{E} = -\vec{\nabla}V$$

Electric Potential: Some comments

If you know V, you will get $\vec{E}$ since $\vec{E} = -\vec{\nabla} V$.

How can one function (V) contain all the information that three independent functions (components of $\vec{E}$) carry?

The three components of $\vec{E}$ are not independent. They are interrelated by the condition $\vec{\nabla} \times \vec{E} = 0$.

The components are related as:

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}, \quad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z}, \quad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

Electric Potential: Some comments

The reference point $\mathcal{O}$ is completely arbitrary. Changing reference point is same as adding a constant C to the potential.

$$\boxed{\bar{V}(r) = -\int_{\mathcal{O}'}^{r} \vec{E}.d\vec{\ell} = -\int_{\mathcal{O}'}^{\mathcal{O}} \vec{E}.d\vec{\ell} - \int_{\mathcal{O}}^{r} \vec{E}.d\vec{\ell} = C + V(r)}$$

where C is the line integral of $\vec{E}$ from the old reference point $\mathcal{O}$ to the new one $\mathcal{O}'$.

Note: adding a constant to the potential will not affect the potential difference between two points

$$\bar{V}(b) - \bar{V}(a) = V(b) - V(a)$$

Since the derivative of a constant is zero : $\vec{\nabla}\bar{V} = \vec{\nabla}V$

All V's, differing only in their choice of reference point correspond to the same field $\vec{E}$.

Choosing a reference point is equivalent to selecting a place where V is to be zero.

Like the electric field, potential also obeys the superposition principle : potential at any point is the sum of the potentials due to all the source charges separately : $V = V_1 + V_2 + \cdots$

Solving for the potential

So far, we have obtained two important laws regarding the electric field

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \times \vec{E} = 0$$

$\vec{E} = -\vec{\nabla} V$

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$$ Poisson's equation

In source free region
(no charge) :$\rho = 0$

$$\boxed{\nabla^2 V = 0}$$ Laplace's equation

$$\vec{\nabla} \times \vec{E} = \vec{\nabla} \times (-\vec{\nabla} V) = 0$$

No condition on V since curl of gradient is always zero. $\vec{\nabla} \times \vec{E} = 0$ permits $\vec{E} = -\vec{\nabla} V$.

We require only one differential equation (Poisson's equation) to determine V, because V is a scalar.

Potential for a localised charge distribution

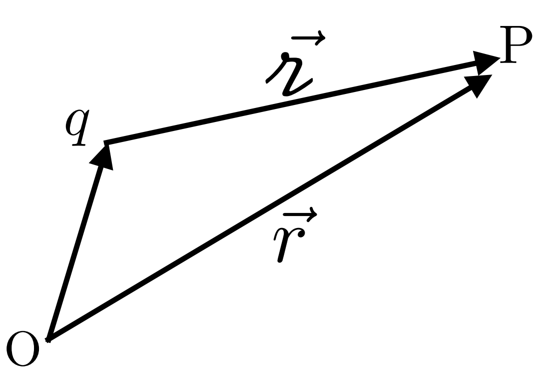
Point charge: The electric field is $\vec{E} = (1/4\pi\epsilon_0)(q/r^2)\hat{r}$ and $d\vec{\ell} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$ so that

$$\vec{E}.d\vec{\ell} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$$

Setting the reference point $\mathcal{O}$ at infinity, the potential of a point charge q at origin is

$$V(r) = -\frac{1}{4\pi\epsilon_0}\int_\infty^r \frac{q}{r'^2}dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\Big|_\infty^r = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

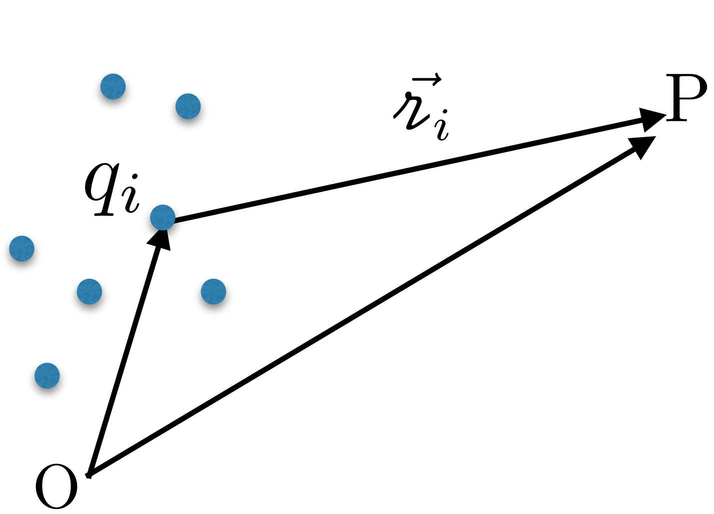
Note : 1. Arbitrariness in choosing the reference point helped here. Choosing infinity as reference point killed the lower limit on the integral.



In general the potential for a point charge q is

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$

Potential for a localised charge distribution

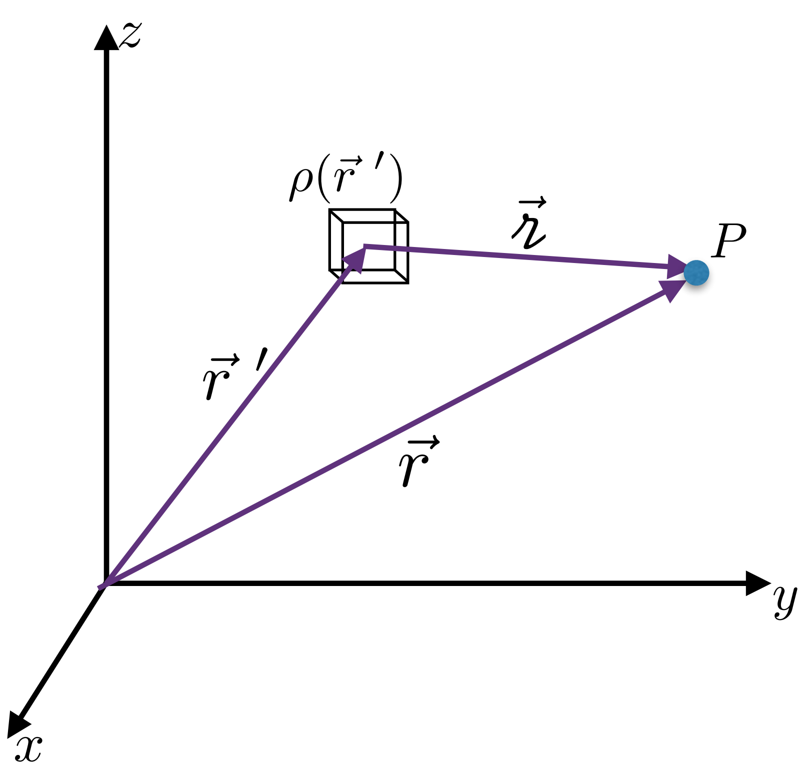


Collection of point charges $$V(r) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$$

Line charge $$V(r) = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\vec{r}\,')}{\mathscr{r}}d\ell'$$

Surface charge $$V(r) = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\vec{r}\,')}{\mathscr{r}}da'$$

Volume charge $$V(r) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}\,')}{\mathscr{r}}d\tau'$$

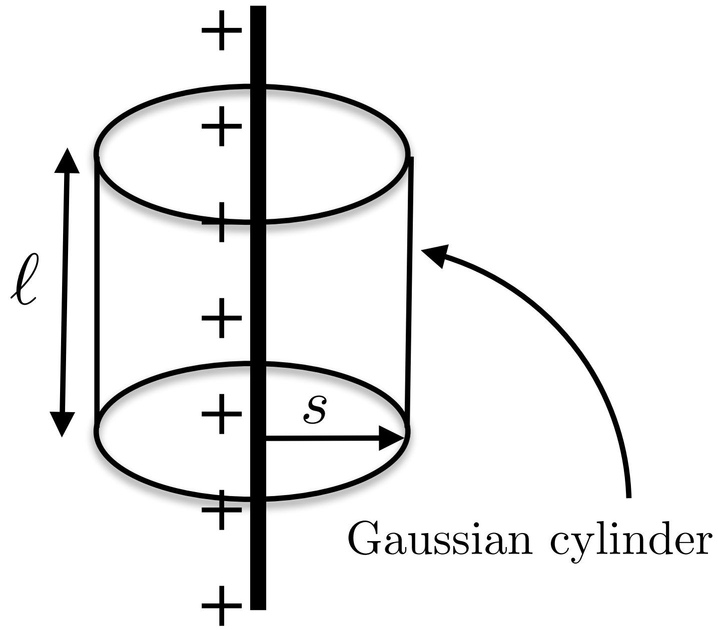


$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$

Remember: These can be regared as soln of Poisson's eqn, provided the ref. pt. is at ∞ where $V(\infty) = 0$.

Example: Potential due to a line charge λ

Field can be evaluated by enclosing the line charge with a Gaussian cylinder of length ℓ and radius s.



The field only depends on distance from the line charge and directed away from it. Contribution to the flux from the top and bottom caps of the cylinder are zero.

$$\int \vec{E}.d\vec{a} = |E| 2\pi s \ell = \frac{\lambda \ell}{\epsilon_0}$$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$$

fixed by reference point

$$\vec{E} = -\vec{\nabla} V \implies V = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s}{l}\right) + \text{Constant}$$

; l = some length scale.

Warning: Unlike the case of point charge, the reference point can not be taken at infinity! Why?

logarithm gets undefined! Choose the reference point at $\frac{s}{l} = 1$ where the potential becomes zero.

$$\therefore\ V = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s}{l}\right) \quad \text{with respect to reference point at } \frac{s}{l} = 1$$

Potential $V = -\frac{\lambda}{2\pi\epsilon_0} \ln(s/l)$

Remember : l = some length scale of the system.

Potential diff^n does not dependent on "l":

$$V_b - V_a = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_b}{l}\right) + \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_a}{l}\right)$$

$$= -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_b}{s_a}\right).$$

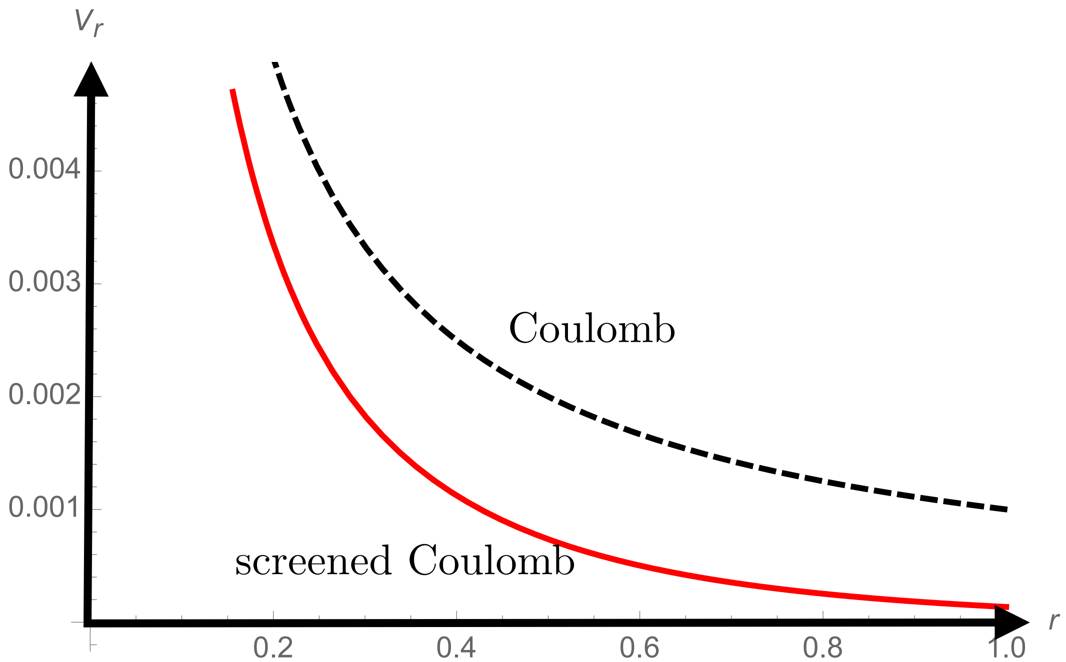
Example: Screened Coulomb potential

Consider the so-called "screened Coulomb potential" of a point charge q that arises, for example, in plasma physics

$$V(r) = \frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r}$$

λ is a constant called screening length.

Determine the charge distribution $\rho(r)$ that produces this potential.



$$\begin{aligned}\vec{E} = -\vec{\nabla}V &= -\frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{e^{-r/\lambda}}{r}\right)\hat{r} \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}\hat{r}\end{aligned}$$

The charge distribution that gives rise to this electric field can be obtained by calculating $\vec{\nabla}.\vec{E}$

$$\begin{aligned}\vec{\nabla}.\vec{E} &= \frac{q}{4\pi\epsilon_0}\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left[\left\{\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right)\right\}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda} + \frac{\hat{r}}{r^2}.\vec{\nabla}\left\{\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}\right\}\right]\end{aligned}$$

using $\vec{\nabla}.(f\vec{A}) = f(\vec{\nabla}.\vec{A}) + \vec{A}.(\vec{\nabla}f)$

Example: Screened Coulomb potential

$$\text{using} \quad \vec{\nabla}.(\hat{r}/r^2) = 4\pi\delta^3(r)$$

$$\begin{aligned}\vec{\nabla}.\vec{E} &= \frac{q}{4\pi\epsilon_0}\left[e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)4\pi\delta^3(\vec{r}) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)\right)\right] \\ &= \frac{q}{4\pi\epsilon_0}\left[e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)4\pi\delta^3(\vec{r}) - \frac{1}{r\lambda^2}e^{-r/\lambda}\right] \equiv \frac{\rho}{\epsilon_0}\end{aligned}$$

$$\boxed{\rho(r) = e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)q\delta^3(\vec{r}) - \frac{q}{4\pi\lambda^2 r}e^{-r/\lambda}}$$

There is a point charge at the origin in addition to an exponentially decaying charge density

Summary

Fundamental quantities in electrostatics : $\rho, \vec{E}, V$.

Began with (i) superposition principle and (ii) Coulomb's law. All else followed from these.