Physics II: Electromagnetism

PH 102

Lecture 8

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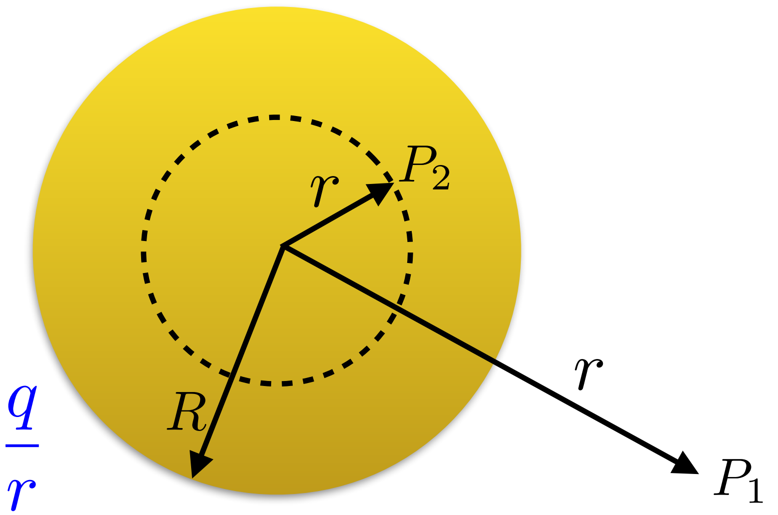
Example: Potential due to uniformly charged sphere

q: total charge distributed over sphere



Recall that: $$\vec{E} = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{r} & r < R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} & r > R \end{cases}$$

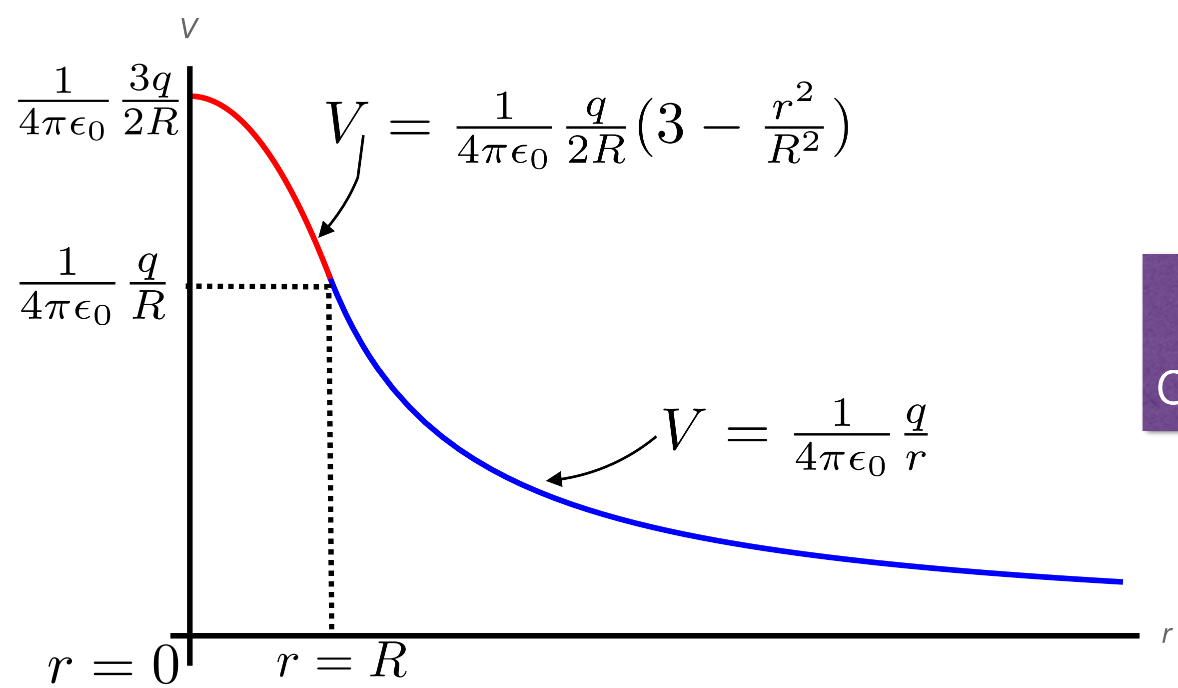
The electric potential at P_1: $$-\int_{\infty}^{r} \vec{E}.d\vec{r}' = -\int_{\infty}^{r} \frac{q}{4\pi\epsilon_0 r'^2} dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

(i.e. r > R)

The electric potential at P_2: $$-\int_{\infty}^{R} \vec{E}.d\vec{r}\,' - \int_{R}^{r} \vec{E}.d\vec{r}\,' = -\int_{\infty}^{R} \frac{q}{4\pi\epsilon_0 r'^2} dr\,' - \int_{R}^{r} \frac{qr\,'}{4\pi\epsilon_0 R^3} dr\,'$$

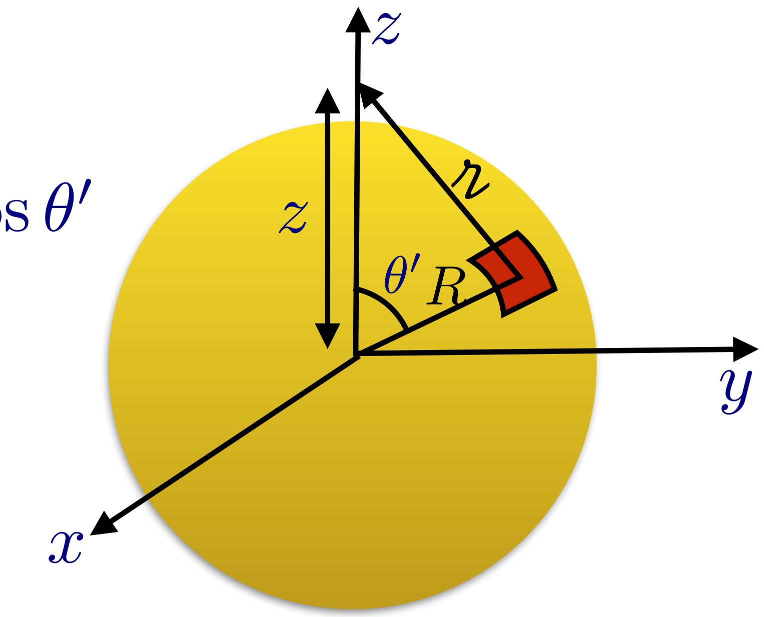
(i.e. r < R)

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{R} - \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\frac{1}{2}(r^2 - R^2) = \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right)$$



Both E & V are continuous at the surface

Potential due to uniformly charged spherical shell



$$V(\mathscr{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\mathscr{r}}da \qquad \text{where } \mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta'$$

$$4\pi\epsilon_0 V(z) = \sigma \int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi} \frac{R^2\sin\theta' d\theta' d\phi'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}$$

$$= 2\pi R^2\sigma \int_{\theta'=0}^{\pi} \frac{\sin\theta' d\theta'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}$$

$$= 2\pi R^2\sigma \left(\frac{1}{Rz}\sqrt{R^2+z^2-2Rz\cos\theta'}\right)\Bigg|_0^{\pi}$$

$$= \frac{2\pi R\sigma}{z}\left(\sqrt{R^2+z^2+2Rz} - \sqrt{R^2+z^2-2Rz}\right)$$

$$= \frac{2\pi R\sigma}{z}\left(\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right)$$

$$V(z) = \frac{R\sigma}{2\varepsilon_0 z}[(R+z)-(z-R)] = \frac{R^2\sigma}{\varepsilon_0 z}, \quad \textit{outside}$$

$$V(z) = \frac{R\sigma}{2\varepsilon_0 z}[(R+z)-(R-z)] = \frac{R\sigma}{\varepsilon_0}, \quad \textit{inside}$$

Note: outside the shell, z>R

Use Gauss's law to find E and then potential V

$$E_{out} \cdot 4\pi z^2 = \frac{\sigma \times 4\pi R^2}{\epsilon_0} \Rightarrow E_{out} = \frac{\sigma R^2}{\epsilon_0 z^2} = -\frac{dV_{out}}{dz}$$

$$V_{out} = -\frac{\sigma R^2}{\epsilon_0}\int_{\infty}^{z}\frac{dz'}{z'^2} = \frac{\sigma R^2}{\epsilon_0 z}$$

Reference point has been chosen at z=infinity

$$E_{in} \cdot 4\pi z^2 = 0 \quad \text{as} \quad Q_{encl} = 0$$

$$\Rightarrow \frac{dV_{in}}{dz} = 0$$

$$V_{in} = Constant(C)$$

At z=R, $V_{in} = V_{out}$

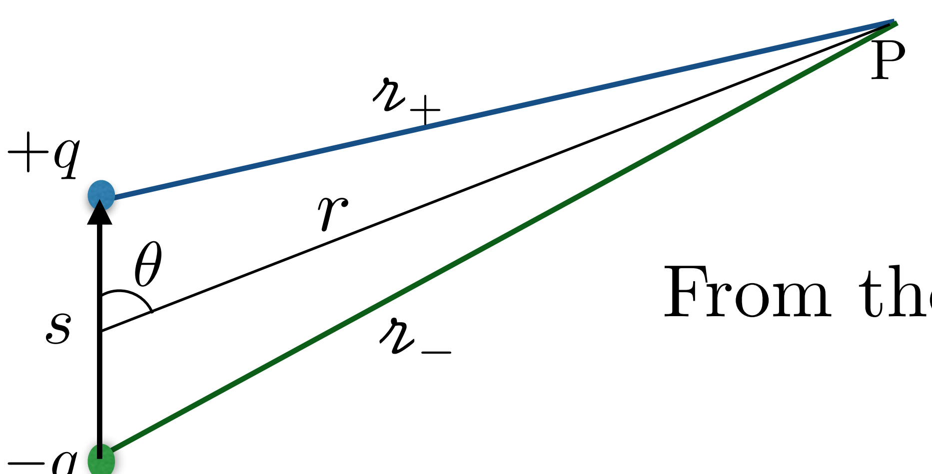
$$\Rightarrow V_{in} = C = \frac{\sigma R}{\epsilon_o}$$

E is discontinuous at the surface; V is continuous

Example: Potential and electric field of a dipole

An electric dipole consists of two equal and opposite charges ($\pm q$) separated by a distance s.

The dipole moment is a vector of magnitude equal to the product of the magnitude of either charge and the distance between them. Conventionally the direction of the dipole moment vector is defined along the direction from negative to positive charge.



$$V(P) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$

From the geometry: (using law of cosines)

$$\begin{aligned} \mathscr{r}_\pm^2 &= r^2 + \left(\frac{s}{2}\right)^2 \mp r s \cos\theta \\ &= r^2\left(1 \mp \frac{s}{r}\cos\theta + \frac{s^2}{4r^2}\right) \end{aligned}$$

negligible

We are interested in the regime $r >> s$

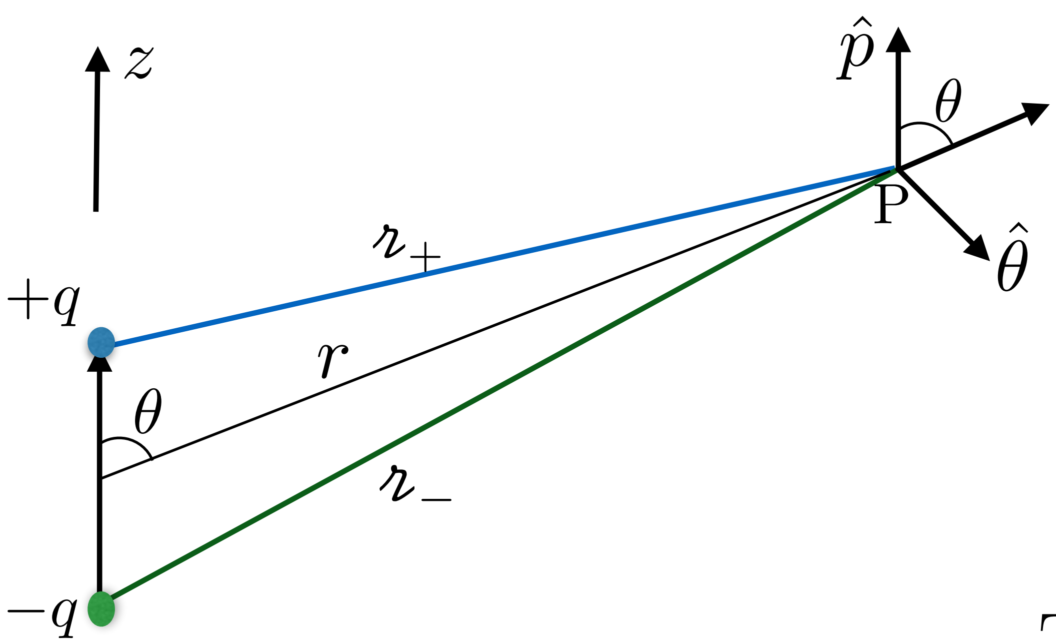
$$\frac{1}{\mathscr{r}_\pm} \simeq \frac{1}{r}\left(1 \mp \frac{s}{r}\cos\theta\right)^{-1/2} \simeq \frac{1}{r}\left(1 \pm \frac{s}{2r}\cos\theta\right)$$

Example: Potential and electric field of a dipole

Thus $\dfrac{1}{\mathscr{r}_+} - \dfrac{1}{\mathscr{r}_-} \simeq \dfrac{s}{r^2}\cos\theta$ and hence $V(P) \simeq \dfrac{1}{4\pi\epsilon_0}\dfrac{qs\cos\theta}{r^2}$

$$\longrightarrow \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{\vec{p}.\hat{r}}{r^2}}$$

Recall: dipole moment $\vec{p} = qs\hat{p}$. Hence $qs\cos\theta = p\cos\theta = \vec{p}.\hat{r}$



Now $\vec{E} = -\vec{\nabla}V$, but we need $\vec{\nabla}$ in spherical polar coordinate

$$\vec{\nabla} \equiv \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Thus
$$\begin{aligned}\vec{E} &= -\frac{p}{4\pi\epsilon_0}\vec{\nabla}\frac{\cos\theta}{r^2} \\ &= -\frac{p}{4\pi\epsilon_0}\left(\hat{r}\frac{\partial}{\partial r}\frac{\cos\theta}{r^2} + \hat{\theta}\frac{1}{r^3}\frac{\partial\cos\theta}{\partial\theta}\right) \\ &= \boxed{\frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})}\end{aligned}$$

Assumes spherical polar and direction of $\vec{p}$ along z

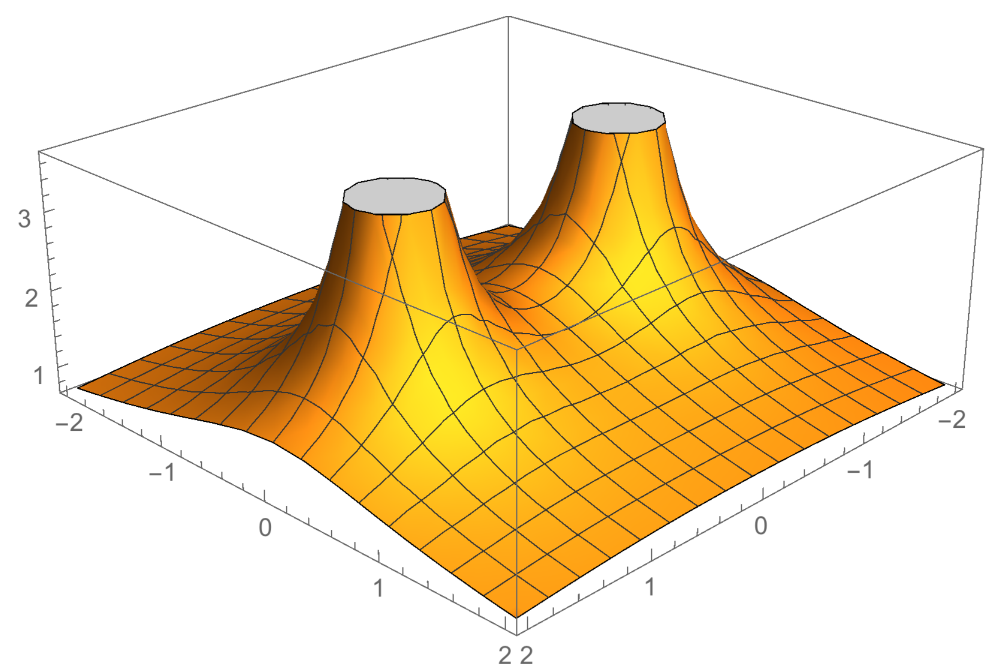
Dipole field falls off as inverse cube of r

How to "see" the potential?

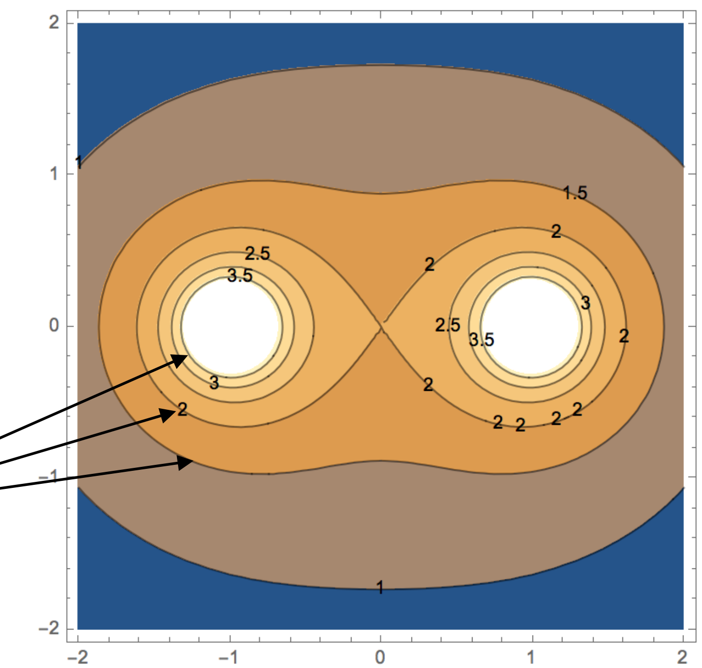
The easiest way to represent the potential is to draw surfaces on which **V** is a constant. We call them equipotential surfaces—surfaces of equal potential (recall level surfaces/curves from L1)

Potential due to 2 identical point charges at (1,0,0) and (-1,0,0)

$$V(x,y) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-1)^2+y^2}} + \frac{1}{\sqrt{(x+1)^2+y^2}}\right)$$



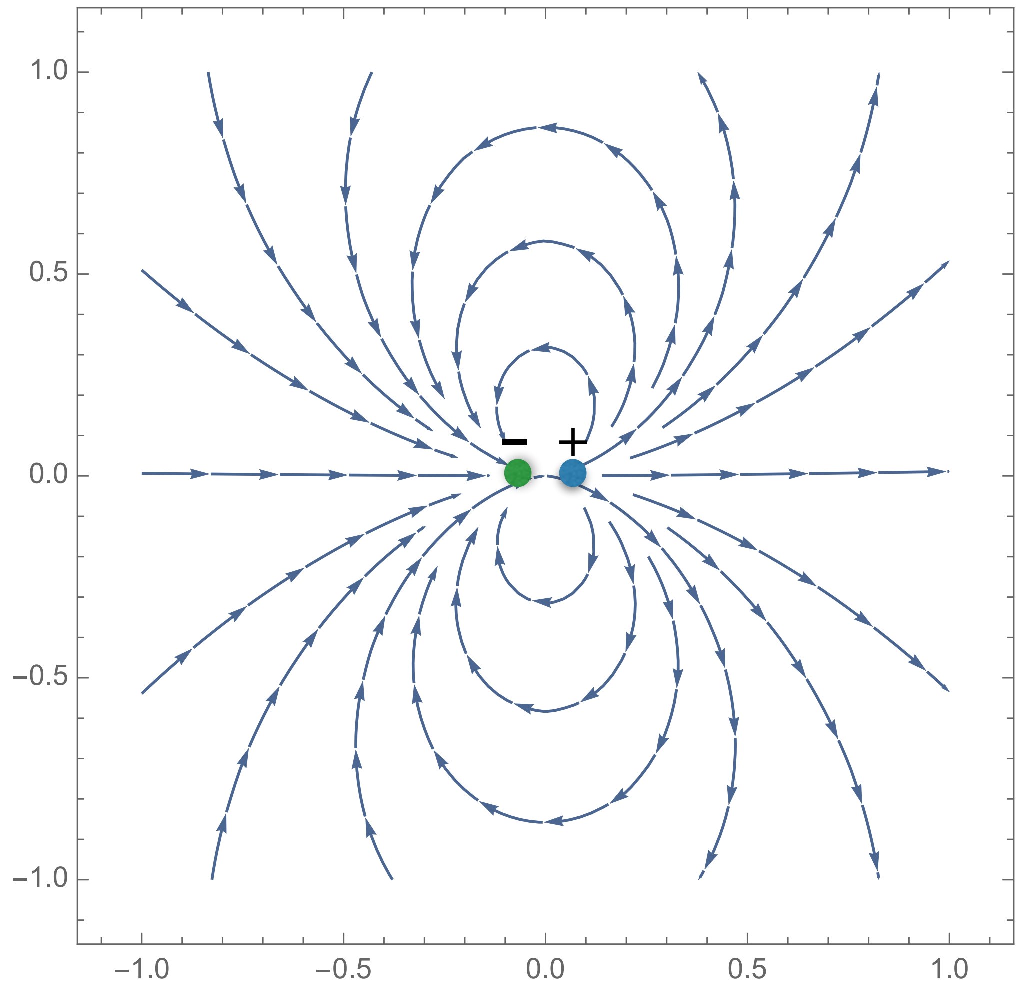
Equipotentials

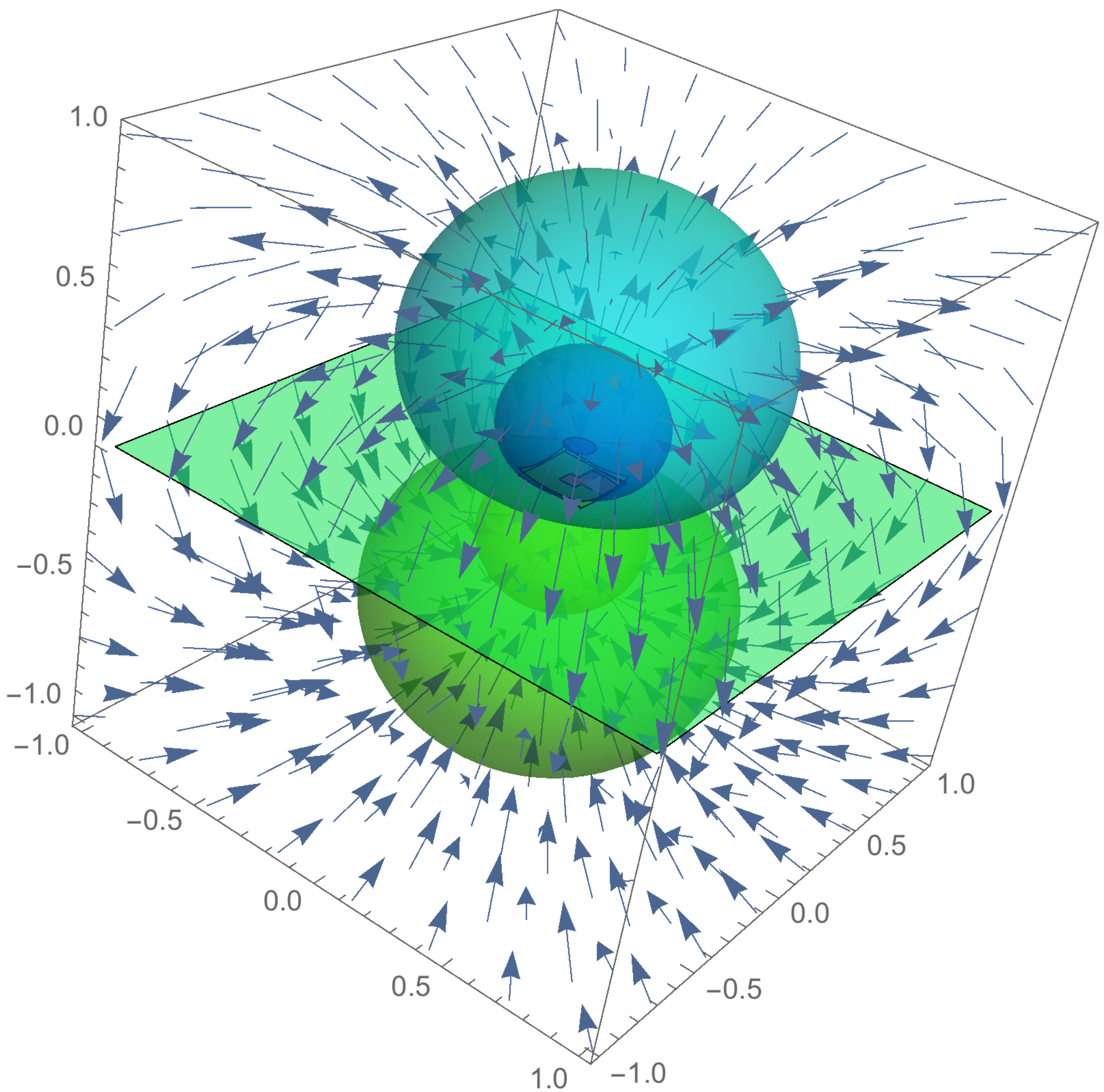


Q. What is the geometrical relationship of the equipotential surfaces to the field lines?

Remember: $E = -\vec{\nabla}V$. The gradient is in the direction of the most rapid change of the potential, and hence is perpendicular to an equipotential surface. If E were not perpendicular to the surface, it would have a component in the surface. The potential would be changing in the surface, but then it wouldn't be an equipotential. The equipotential surfaces must then be everywhere at right angles to the electric field lines.

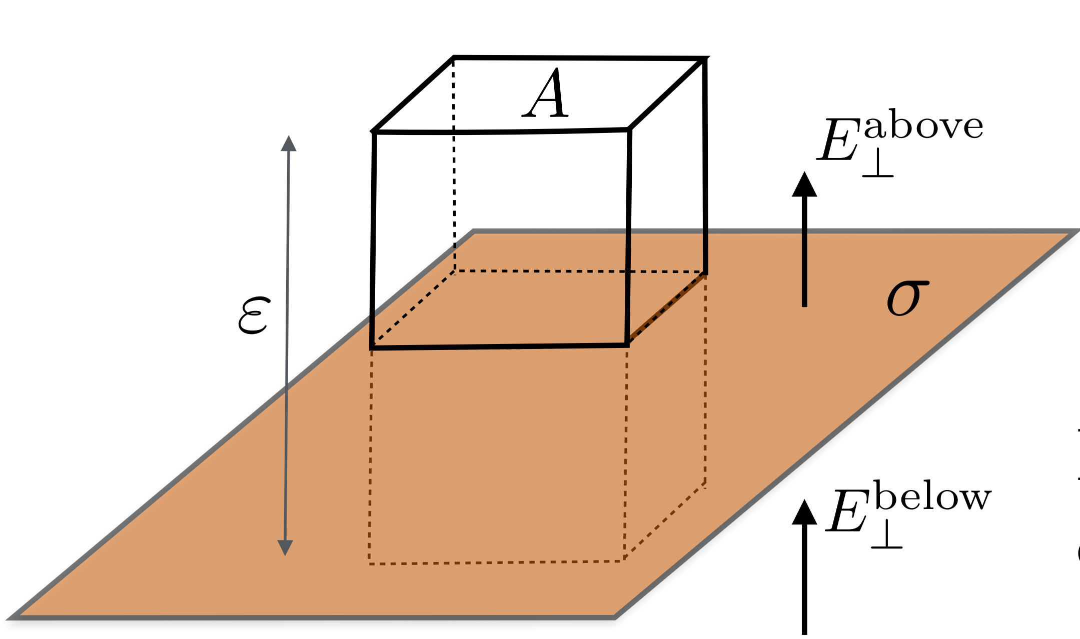
Dipole: Field lines and Equipotentials





Electrostatic Boundary Conditions

We have noticed that whenever there is a surface charge, the electric field is discontinuous across the surface but the potential is continuous.

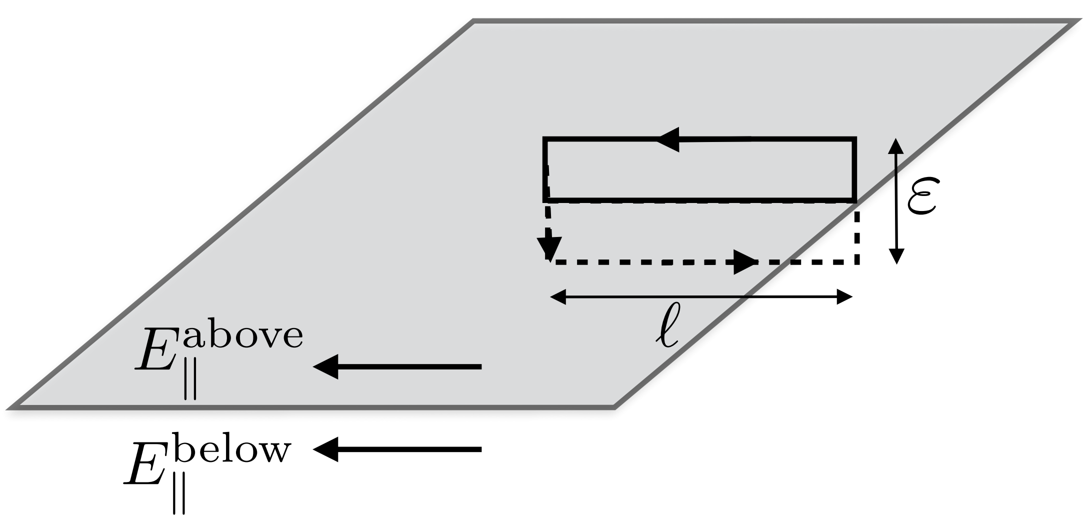


$$\oint_{\mathcal{S}} \vec{E}.d\vec{a} = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{1}{\epsilon_0}\sigma A$$

As $\varepsilon \to 0$ $\quad E_\perp^{\text{above}} A - E_\perp^{\text{below}} A = \frac{1}{\epsilon_0}\sigma A$

Normal component of the electric field is discontinuous by an amount

$$\boxed{E_\perp^{\text{above}} - E_\perp^{\text{below}} = \frac{\sigma}{\epsilon_0}}$$



Since $\vec{E}$ is conservative field, we have $\oint \vec{E}.d\vec{\ell} = 0$ always. Applying this to the sides as $\varepsilon \to 0$: $E_\parallel^{\text{above}}\ell - E_\parallel^{\text{below}}\ell = 0$

$\therefore$ for the tangential component

$$\boxed{E_\parallel^{\text{above}} = E_\parallel^{\text{below}}}$$

Electrostatic Boundary Conditions

The boundary conditions on $\vec{E}$ can be combined to a single formula

$$\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{n} \qquad \hat{n}\text{: unit vector} \perp \text{to surface}$$

Potential is continuous across any boundary: $V_{\text{above}} - V_{\text{below}} = \int_a^b \vec{E}.d\vec{\ell}$

The path length shrinks to zero on the boundary and hence

$$V_{\text{above}} = V_{\text{below}}$$

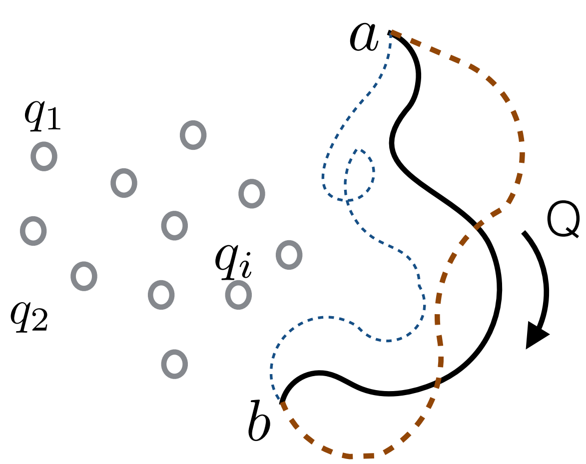
But, since $\vec{E} = -\vec{\nabla}V$, gradient of V inherits the discontinuity

$$\vec{\nabla}V_{\text{above}} - \vec{\nabla}V_{\text{below}} = -\frac{\sigma}{\epsilon_0}\hat{n}$$

$$\text{or,} \quad \frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

where $\frac{\partial V}{\partial n} = \vec{\nabla}V.\hat{n}$ is the normal derivative of of V
(rate of change in the direction $\perp$ to surface)

Energy of a point charge in Electric Field



Let us have electric field $\vec{E}$ and potential V due to a stationary configuration of source charges

The electrostatic force field $\vec{F}$ on a test charge Q is conservative since

$$\vec{\nabla} \times \vec{F} = Q\vec{\nabla} \times \vec{E} = 0$$

Question: How much work needs to be done in taking Q from a to b?

Work done by $\vec{F}$

$$W = \int_a^b \vec{F}.d\vec{\ell} = -Q\int_a^b \vec{E}.d\vec{\ell} = Q[V(b) - V(a)]$$

independent of path, depends on end points

at any point along the path, the force on Q is F=QE. The force you need to exert in opposition to this electrical force is -QE.

$$V(b) - V(a) = \frac{W}{Q}$$

Potential difference between a and b is the work per unit charge required to carry a particle from a to b

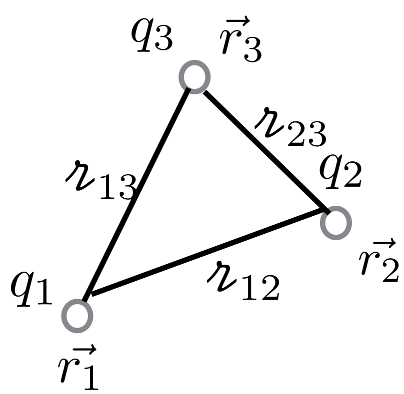
$$W = Q[V(r) - V(\infty)]$$

potential energy per unit charge : potential

$\rightarrow W = QV(r)$ if reference at ∞

Energy of a point charge distribution

Question: How much work is needed to assemble an entire collection of point charges?



No work to bring in the first charge q_1 at $\vec{r_1}$

Total work done in placing the second charge $W_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{\mathscr{r}_{12}}\right)$

where $\mathscr{r}_{12} = |\vec{r}_1 - \vec{r}_2|$

Work done in placing the third charge $W_3 = \frac{1}{4\pi\epsilon_0} q_3 (\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}})$

Total work done in placing the third charge q_3 at $\vec{r}_3$

$$W = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_2 q_3}{\mathscr{r}_{23}}\right)$$

Generalised formula for n charges

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$= \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

Intentionally counting twice and dividing by 2.
However, we must avoid i=j though.

Energy of a point charge distribution

You can also write the energy as:

$$W = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$= \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{j\neq i}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{\mathscr{r}_{ij}}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i).$$

where $V(\vec{r}_i)$ is the scalar potential experienced by the i-th point charge due to other point charges.

W represents the amount of work needed to assemble a configuration of point charges.

$\rightarrow$ it also is the amount of work we get back by dismantling the system

Remember: The factor of 1/2 was introduced because in the double summation, we have counted all pairs of point charges twice.

Energy of a continuous charge distribution

Charge distribution with volume charge density ρ

Each volume element $d\tau$ contains charge $\rho d\tau$.

Generalising the previous case:

$$W = \frac{1}{2}\int\int \frac{1}{4\pi\epsilon_0}\frac{\rho(1)\rho(2)}{\mathscr{r}_{12}}d\tau_1 d\tau_2$$

$$= \frac{1}{2}\int \rho(1)\int \frac{1}{4\pi\epsilon_0}\frac{\rho(2)}{\mathscr{r}_{12}}d\tau_2 d\tau_1$$

potential $V(1)$ at (1)

since (2) no longer appears

$$= \frac{1}{2}\int \rho V d\tau$$

The energy of the charge $\rho d\tau$ is the product of this charge and the potential at the same point. The total energy therefore is the integral over $V\rho d\tau$

But there is still the factor $\frac{1}{2}$!

Energy of a continuous charge distribution

The $\frac{1}{2}$ is still required because we are counting energies twice. The mutual energies of two charges is the charge of one times the potential at it due to the other. Or, it can be taken as the second charge times the potential at it from the first. Thus for two point charges we could write:

$$\begin{aligned} W &= q_1 V(1) = q_1 \frac{1}{4\pi\epsilon_0} \frac{q_2}{\mathscr{r}_{12}} \\ \text{or} \quad W &= q_2 V(2) = q_2 \frac{1}{4\pi\epsilon_0} \frac{q_1}{\mathscr{r}_{12}} \end{aligned}$$

But, note: we could also write

$$W = \frac{1}{2}[q_1 V(1) + q_2 V(2)]$$

The integral in the previous slide corresponds to the sum of both terms in the brackets of the above equation. That is why we need the factor $\frac{1}{2}$.

Energy of a continuous charge distribution $= \frac{1}{2}\int \rho V\, d\tau$

Corresponding integrals for line and surface charges will be $\int \lambda V\, d\ell$ and $\int \sigma V\, da$.

Recall that $\rho = \epsilon_0 \vec{\nabla}.\vec{E}$ and $\vec{E} = -\vec{\nabla} V$. Therefore $\rho = -\epsilon_0 \nabla^2 V$. So that we can write

$$W = -\frac{\epsilon_0}{2}\int V \nabla^2 V\, d\tau$$

$$\begin{aligned} V\nabla^2 V &= V\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) \\ &= \frac{\partial}{\partial x}\left(V\frac{\partial V}{\partial x}\right) - \left(\frac{\partial V}{\partial x}\right)^2 + \frac{\partial}{\partial y}\left(V\frac{\partial V}{\partial y}\right) - \left(\frac{\partial V}{\partial y}\right)^2 + \frac{\partial}{\partial z}\left(V\frac{\partial V}{\partial z}\right) - \left(\frac{\partial V}{\partial z}\right)^2 \\ &= \vec{\nabla}.(V\vec{\nabla} V) - (\vec{\nabla} V).(\vec{\nabla} V). \end{aligned}$$

Therefore $\quad W = \frac{\epsilon_0}{2}\int (\vec{\nabla} V).(\vec{\nabla} V) d\tau - \frac{\epsilon_0}{2}\int \vec{\nabla}.(V\vec{\nabla} V) d\tau$

But using Gauss's theorem $\quad \int_{vol.} \vec{\nabla}.(V\vec{\nabla} V) d\tau = \int_{surf.} (V\vec{\nabla} V).\hat{n} da.$

Energy of a continuous charge distribution

Hence $W = \frac{\epsilon_0}{2}\left(\int_{\text{vol}} (\vec{\nabla}V).(\vec{\nabla}V)d\tau - \int_{\text{surf}} (V\vec{\nabla}V).\hat{n}da\right)$.

Evaluate surface integral in the case when surface goes to infinity.

Why?

So that volume integral becomes integral over all space

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} (\vec{\nabla}V).(\vec{\nabla}V)d\tau = \frac{\epsilon_0}{2}\int_{\text{all space}} \vec{E}.\vec{E}d\tau$$

How?

Take a spherical surface of enormous radius R with centre at the origin

Far away from charges, V varies as $1/R$ and $\vec{\nabla}V$ as $1/R^2$, but surface area increases as R^2

Surface integral falls off as $(1/R)(1/R^2)R^2 \sim (1/R)$ as R increases

Including all space ($R \rightarrow \infty$) means the surface integral $\rightarrow 0$