# Physics II: Electromagnetism

## PH 102

## Lecture 9

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# Brief recap

**Energy of a point charge distribution :**

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$= \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

**Can also be written as :**

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\left(\sum_{j\neq i}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{\mathscr{r}_{ij}}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i).$$

**Energy of a continuous charge distribution :**

$$= \frac{1}{2}\int \rho V\, d\tau$$

Recall that $\rho = \epsilon_0 \vec{\nabla}.\vec{E}$ and $\vec{E} = -\vec{\nabla}V$. Therefore $\rho = -\epsilon_0 \nabla^2 V$. So that we can write

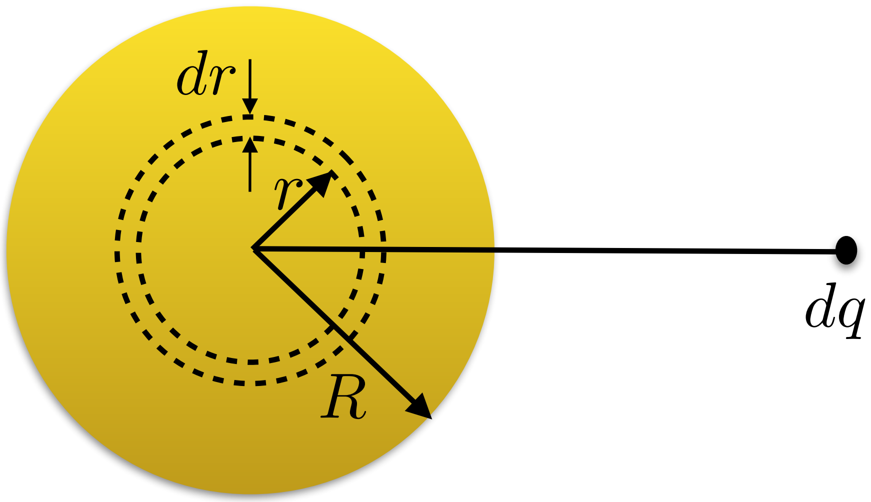
$$W = -\frac{\epsilon_0}{2}\int V\nabla^2 V\, d\tau$$

$$W = \frac{\epsilon_0}{2}\left(\int_{\text{vol}}(\vec{\nabla}V).(\vec{\nabla}V)d\tau - \int_{\text{surf}}(V\vec{\nabla}V).\hat{n}da\right) \longrightarrow \frac{\epsilon_0}{2}\int_{\text{all space}}(\vec{\nabla}V).(\vec{\nabla}V)d\tau = \frac{\epsilon_0}{2}\int_{\text{all space}}\vec{E}.\vec{E}d\tau$$

# Example: Energy of a uniformly charged solid sphere

Solid sphere of radius $R$ and charge $q$

Imagine that we assemble the sphere by building up a succession of thin spherical layers of infinitesimal thickness. At each stage we gather a small amount of charge and put it in a thin layer from $r$ to $r + dr$




Continue the process until we arrive at the final radius $R$

Suppose $q_r$ is the charge of the sphere when it has been built upto radius $r$

Work done in bringing a charge $dq$ to it is $dW = dqV = dq\dfrac{1}{4\pi\epsilon_0}\dfrac{q_r}{r}$

If $\rho$ is the charge density then

$$\begin{aligned} q_r &= \rho\,\frac{4}{3}\pi r^3 \\ \text{and}\quad dq &= \rho\, 4\pi r^2 dr \\ \implies dW &= \frac{4\pi\rho^2 r^4 dr}{3\epsilon_0} \end{aligned}$$

$$W = \frac{4\pi\rho^2}{3\epsilon_0}\int_0^R r^4 dr = \frac{4\pi\rho^2 R^5}{15\epsilon_0} = \frac{4\pi}{15\epsilon_0}\frac{q^2}{(\frac{4}{3}\pi R^3)^2}R^5 = \frac{3}{5}\frac{q^2}{4\pi\epsilon_0 R}$$

Square of

$\therefore$ Energy is proportional to total charge and inversely proportional to radius

# Example: Energy of a uniformly charged solid sphere

Alternate method 1: By using $W = \frac{\epsilon_0}{2}\int_{\text{all space}} \vec{E}.\vec{E}\, d\tau$

We have seen that $\vec{E} = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{r} & r < R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} & r > R \end{cases}$

$$
\begin{aligned}
W &= \frac{\epsilon_0}{2}\left(\frac{1}{4\pi\epsilon_0}\right)^2 q^2 \left[\int_0^R \left(\frac{r}{R^3}\right)^2 4\pi r^2 dr + \int_R^\infty \frac{1}{r^4}\, 4\pi r^2 dr\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{1}{R^6}\left(\frac{r^5}{5}\right)\Big|_0^R + \left(-\frac{1}{r}\right)\Big|_R^\infty\right] = \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left(\frac{1}{R}+\frac{1}{5R}\right) = \frac{3}{5}\frac{q^2}{4\pi\epsilon_0 R}
\end{aligned}
$$

Alternate method 2:

You can also use $W = \frac{1}{2}\int \rho\, V\, d\tau$ with $V = \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right)$

Alternate method 3: use $W = \frac{\epsilon_0}{2}\left(\int_{\text{vol}} (\vec{\nabla}V).(\vec{\nabla}V) d\tau - \int_{\text{surf}} (V\vec{\nabla}V).\hat{n}da\right)$

and check what happens if you evaluate the surface integral at infinity

# Where is the energy located?

Who cares?

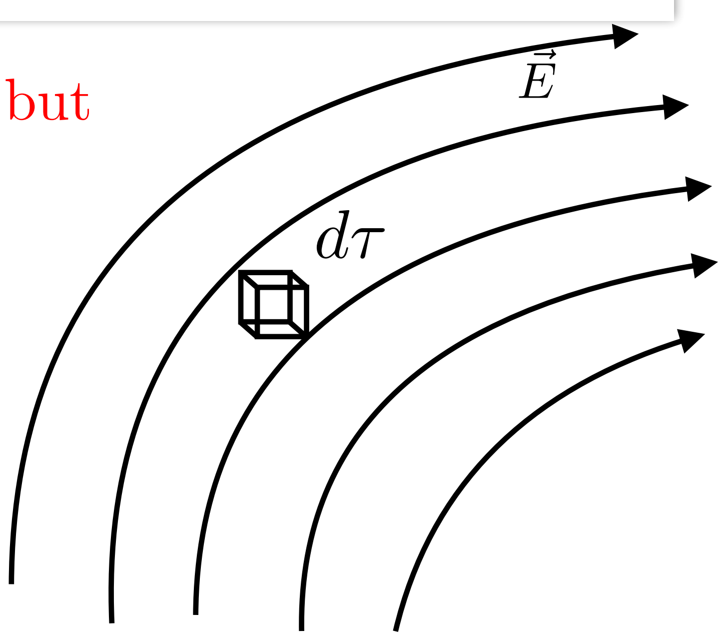
If there is a pair of interacting charges, the combination has certain energy.

Do we need to say the energy is located at one of the charges or the other, or at both, or in between?

In case of electrostatics, it is really hard to answer. Is it stored in the field as $W = \frac{\epsilon_0}{2}\int E^2 d\tau$ may suggest or is it stored in the charges as $W = \frac{1}{2}\int \rho V d\tau$ implies?

It is best to think that the energy is located in space where the electric field is. Define energy density $w = \frac{\epsilon_0 |\vec{E}|^2}{2}$ such that a small volume $d\tau$ will contain electrostatic energy $w \; d\tau$

(Accelerated charges radiate. Also, when light travels, they carry energy but there is no charge $\rightarrow$ Energy is in the field.)




Each volume element $d\tau = dxdydz$ in an electric field contains the energy $(\epsilon_0/2)|\vec{E}|^2 d\tau$

# Self energy : Point charge

Electric field of point charge placed at origin : $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\hat{r}$

Electrostatic energy $W = \dfrac{\epsilon_0}{2}\displaystyle\int E^2\, d\tau$

$$
\begin{aligned}
&= \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \int_{r=0}^{\infty} \frac{1}{r^4}\, 4\pi r^2 dr \\
&= \frac{q^2}{8\pi\epsilon_0} \int_{r=0}^{\infty} \frac{1}{r^2} dr \\
&= -\frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{r}\right)\Big|_{r=0}^{r=\infty} \rightarrow \text{Diverges}
\end{aligned}
$$

There is an infinite amount of energy in the field of a point charge!!

The idea of locating the energy in the field is inconsistent with the existence of point charges. One way out is to say that elementary charges, like the electron, are not point charges but are really small distribution of charges.

Alternatively, there is something wrong in our theory of electricity at very small distances. These difficulties have never been overcome, they exist to this day.

# Interaction energy of 2 point charges: Superposition principle

Take two charges $q_1$, $q_2$ at $\vec{r}_1$, $\vec{r}_2$ respectively.

Electric field at any point $\vec{r}$: $\vec{E}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})$

Where $\vec{E}_1$ and $\vec{E}_2$ are due to $q_1$ and $q_2$ respectively.

Electrostatic energy :
$$\begin{aligned} W_{\text{tot}} &= \frac{\epsilon_0}{2}\int |\vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})|^2 \, d\tau \\ &= \frac{\epsilon_0}{2}\int |\vec{E}_1(\vec{r})|^2 d\tau + \frac{\epsilon_0}{2}\int |\vec{E}_2(\vec{r})|^2 d\tau + \frac{\epsilon_0}{2}\int 2\vec{E}_1(\vec{r}).\vec{E}_2(\vec{r}) d\tau \\ &= W_1 + W_2 + \epsilon_0 \int \vec{E}_1(\vec{r}).\vec{E}_2(\vec{r}) d\tau \end{aligned}$$

Cross term

Because electrostatic energy is quadratic in the field, it does not obey the superposition principle!

The electrostatic energy of a compound system is not the sum of the energies of its parts considered separately!

Interaction energy :
$$W_{\text{int}} = \epsilon_0 \int \vec{E}_1(\vec{r}).\vec{E}_2(\vec{r}) \, d\tau = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{\mathscr{r}}$$

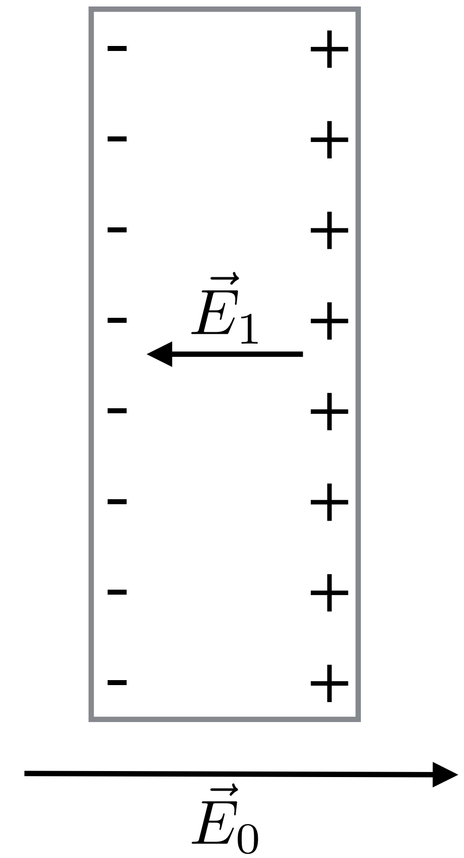
Note: the result is expected but the integral is not easy to perform. Try to do it!!

# Conductors

- Electrical conductor is a solid that contains many free electrons. Electrons can move around freely in a metal but they can not leave the surface. A "perfect conductor" is defined as a material with <u>infinite supply of free charges</u>.

- Any electric field will set large number of free electrons into motion creating current. But in electrostatics, we will not consider such situation.

- An external field $\vec{E}_0$ will separate the positive and negative charges inside the metal and they pile up on two opposite sides. These induced charges produce a field of their own $\vec{E}_1$, which is in opposite direction to $\vec{E}_0$




- Field of the induced charges tends to cancel the original field. Charges will continue to flow until the resultant field inside the conductor is precisely zero. The whole process is practically instantaneous. The only electrostatic situation is that the field is zero everywhere inside.

- $\rho = 0$ inside a conductor: Follows from Gauss's law $\vec{\nabla}.\vec{E} = \rho/\epsilon_0$

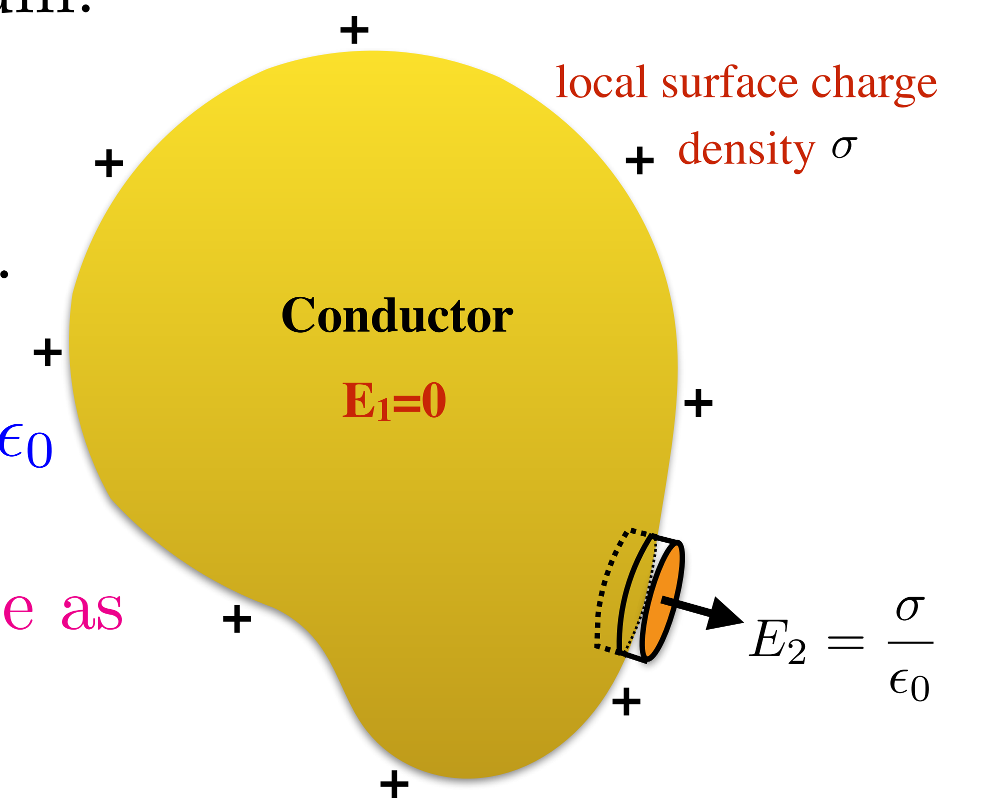
- A conductor is an equipotential and its surface is an equipotential surface, since electric field is zero everywhere, gradient of $V$ should also be zero implying $V$ is constant.

# Conductors

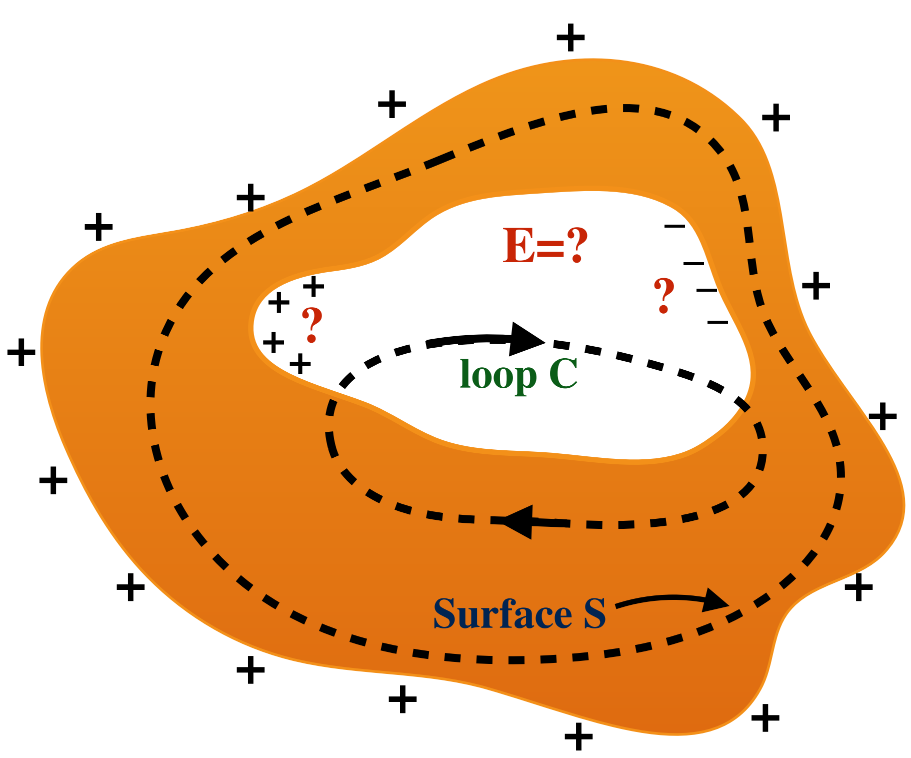
- Any net charge resides on the surface.

- Though there can be charges on the surface, the electric field remains perpendicular to the surface. Because, if $\vec{E}$ had a tangential component then charges on the surface would flow destroying the equilibrium.

- To find $\vec{E}$ outside a conductor, chose a Gaussian surface half inside the conductor and half outside. if $A$ is the area enclosed by the cylinder on the surface, then the flux is $EA = \sigma A/\epsilon_0 \implies E = \sigma/\epsilon_0$




- Note: only contribution to flux is from the top face as the field inside is zero.




There can not be any field inside a cavity within a conductor, nor any charges on the inside surface.

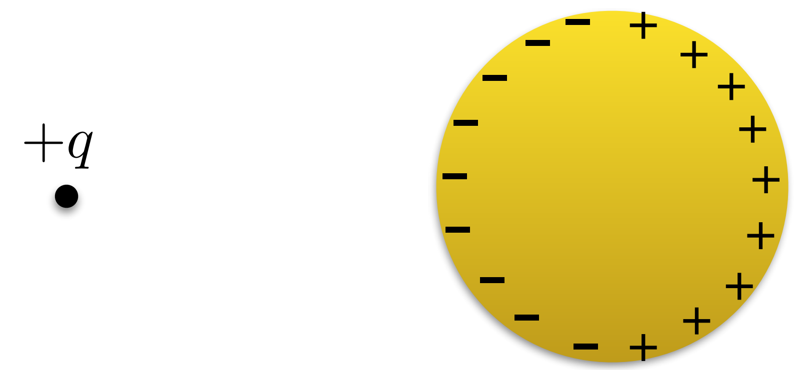
$\longrightarrow$ Principle of 'shielding' electrical equipments

Faraday cage

# Conductors

**Induced charges:**

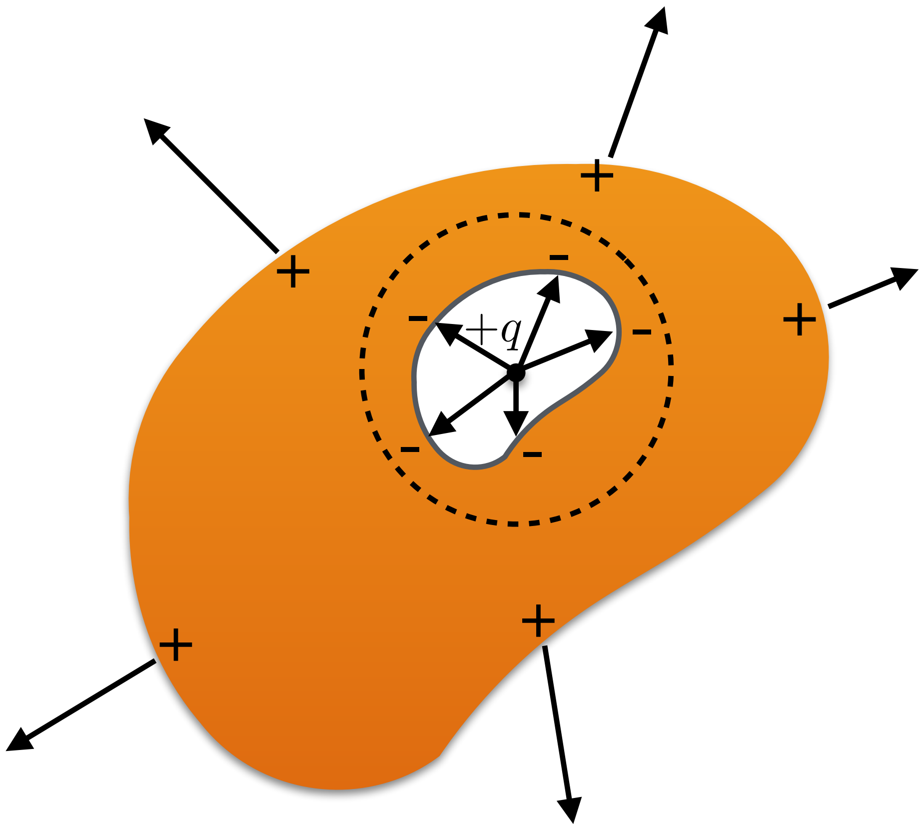
A charge $+q$ near an uncharged conductor will attract each other.




Suppose a charge $+q$ is placed inside the cavity within a conductor.

Will the world outside know about it?

Since no field can enter the conductor, how can it tell the outside world about its presence?




For the Gaussian surface inside the conductor
$\oint \vec{E}.d\vec{a} = 0 \implies Q_{\text{enc}} = 0.$

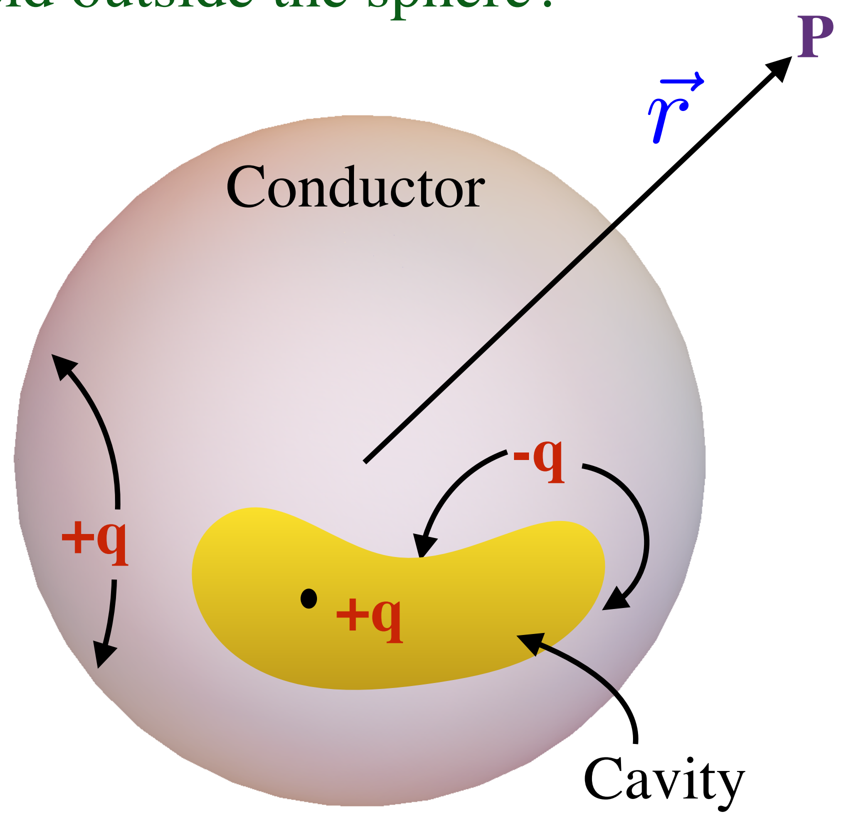
But $Q_{\text{enc}} = q + q_{\text{induced}} \implies q_{\text{induced}} = -q$

Negative charges $(-q)$ are induced on the inner surface and positive charges $(+q)$ go to the outer surface

A Gaussian surface outside the conductor will yield a surface integral corresponding to an enclosed charge $q$

# Example

An uncharged spherical conductor centred at the origin has a cavity of some weird shape caved out of it. Somewhere within the cavity lies a charge q. What is the field outside the sphere?




The answer does not depend on the shape of the cavity and the location of the charge.

The answer always is $$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

The conductor conceals from us all the info concerning the nature of the cavity, revealing only total charge it contains.

The charge +q induces an opposite charge -q on the wall of the cavity, which distributes in such a way that its field cancels that of +q, for all points exterior to cavity.

Since the conductor carries no net charge, this leaves +q to distribute itself uniformly over the surface of the sphere.

# Surface charge and force on a conductor

In the presence of an electric field, a surface charge will experience a force, the force per unit area is :

$$\vec{f} = \sigma \vec{E}$$

But, recall, the electric field is discontinuous at a surface charge $\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{n}$

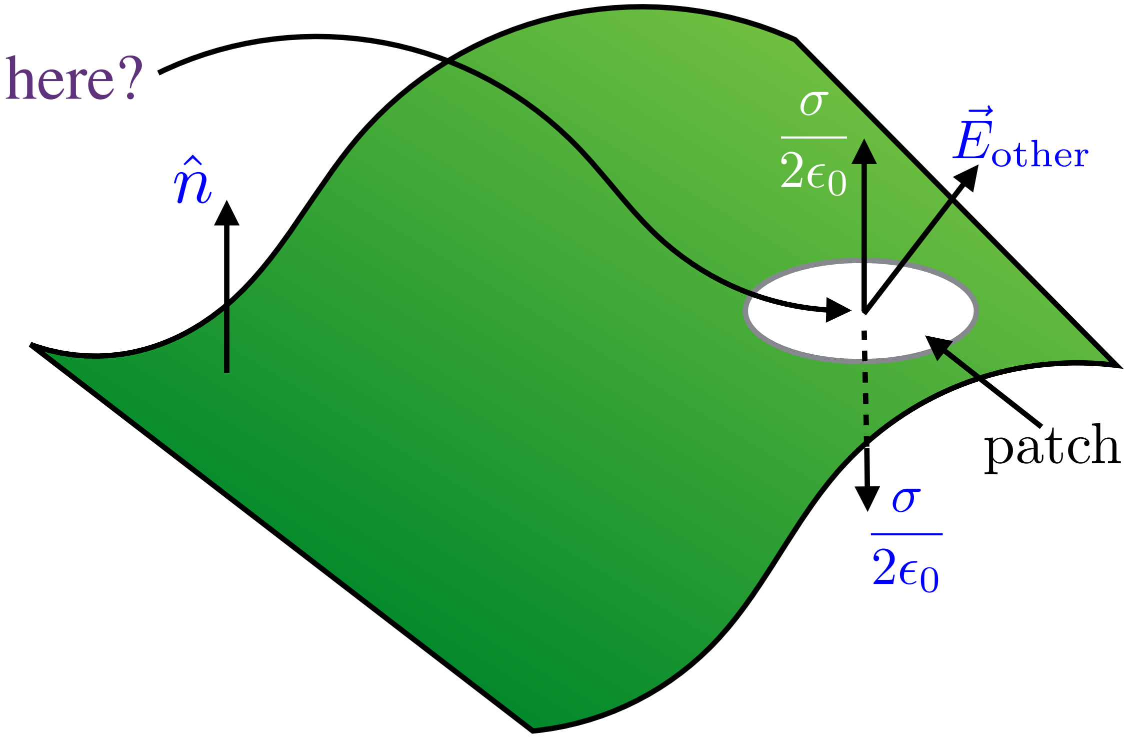
Question: In order to calculate the force, what should we take as the electric field? $E_{\text{above}}$, $E_{\text{below}}$ or something in between?

We should take the average of the two: $\vec{f} = \sigma \vec{E}_{\text{average}} = \frac{1}{2}\sigma(\vec{E}_{\text{above}} + \vec{E}_{\text{below}}).$

WHY?

# Surface charge and force on a conductor

The question is what should we take as the electric field here?




Focus attention on a tiny white patch on the surface surrounding the point where we want the electric field.

Make the patch small enough such that it is essentially flat and surface charge is uniform.

Hence the total field is: $\vec{E} = \vec{E}_{\text{patch}} + \vec{E}_{\text{other}}$

Patch can not exert a force on itself.
Force on the patch is due exclusively to $\vec{E}_{\text{other}}$

Other region of the surface as well as extra sources that may be present!

But note that there is no discontinuity for $\vec{E}_{\text{other}}$

The discontinuity we are talking about is due entirely to the charge on the patch, which puts out a field $\sigma/2\epsilon_0$ on either side. Thus:

$$\begin{aligned} \vec{E}_{\text{above}} &= \vec{E}_{\text{other}} + \frac{\sigma}{2\epsilon_0}\hat{n} \\ \vec{E}_{\text{below}} &= \vec{E}_{\text{other}} - \frac{\sigma}{2\epsilon_0}\hat{n} \end{aligned} \implies \vec{E}_{\text{other}} = \frac{1}{2}(\vec{E}_{\text{above}} + \vec{E}_{\text{below}}) = \vec{E}_{\text{average}}$$

**Averaging is just a device to remove the contribution of the surface charge from the patch itself.**

# Surface charge and force on a conductor

The field inside a conductor is zero, boundary condition requires that the field immediately outside is

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n} \qquad \text{and in terms of potential,} \qquad \sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$

Therefore the average field is $\vec{E}_{\text{average}} = \frac{1}{2}(\vec{E}_{\text{above}} + \vec{E}_{\text{below}}) = \frac{1}{2}(\vec{E}_{\text{above}} + 0) = \frac{\sigma}{2\epsilon_0}\hat{n}$

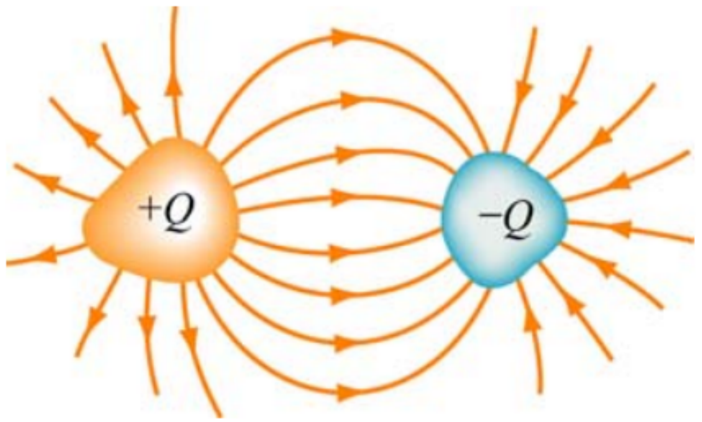
Therefore the force per unit area $\vec{f} = \sigma \vec{E}_{\text{average}} = \frac{1}{2\epsilon_0}\sigma^2\hat{n}$

This amounts to an outward electrostatic pressure $P = \frac{\epsilon_0}{2}E^2$

P is force per unit area;
Replace $\sigma$ in terms of $\vec{E}$ from first equation

# Capacitance: Definitions and all that…

**A capacitor is a device which stores electric charge.**




**Capacitors vary in shape and size**

**basic configuration: two conductors carrying equal but opposite charges**

**What determines how much charge is on the plates of a capacitor for a given voltage?**

**Experiments show that the quantity of charge Q on a capacitor is linearly proportional to the potential difference between the conductors; that is,** $Q \propto \Delta V$

**The proportionality constant C depends on the shape and separation of the conductors.**

Capacitance: $C = \dfrac{Q}{\Delta V}$

**Unit: 1 Farad = 1 F=1C/V**

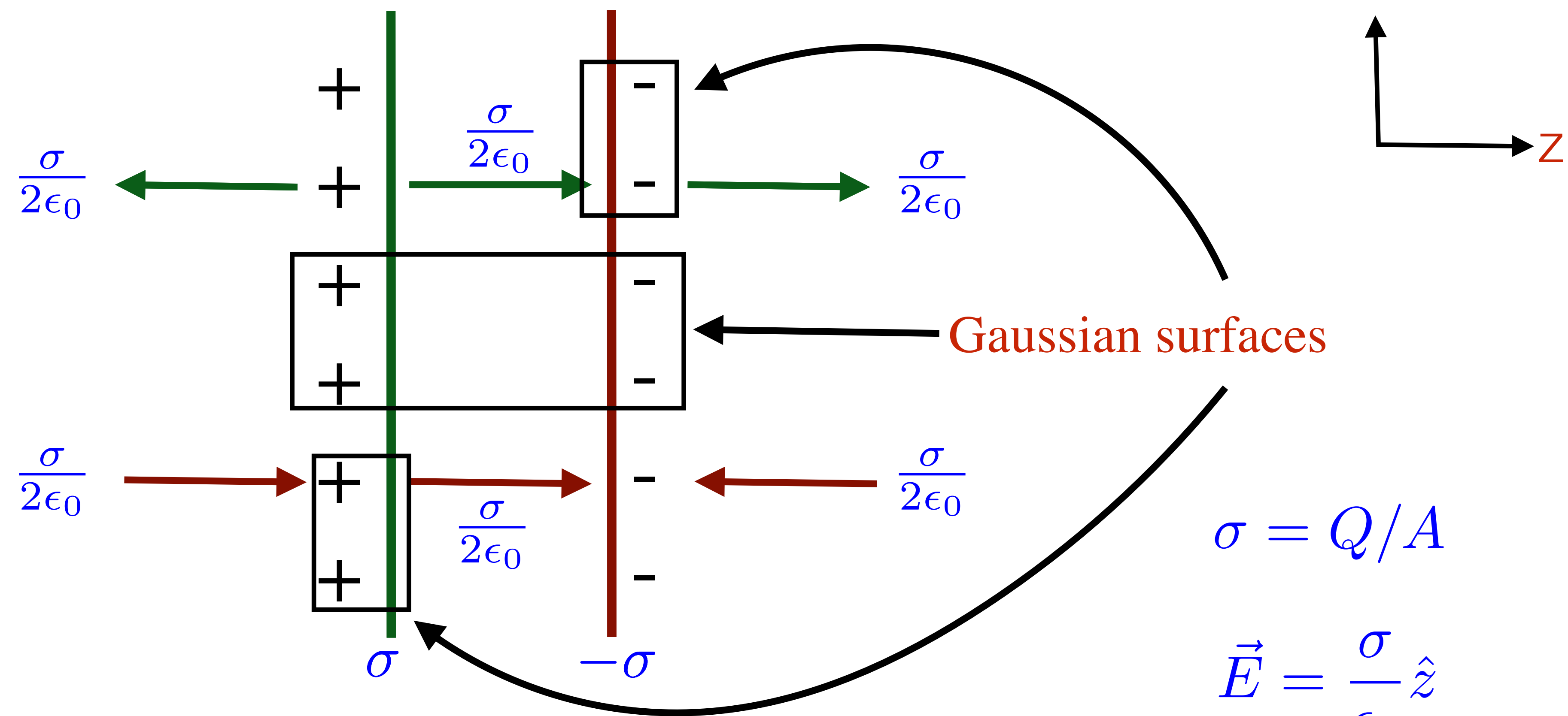
$$\Delta V = V_{+} - V_{-} = -\int_{-}^{+} \vec{E} \, . \, d\vec{l}$$

Remember: Q is the charge of positive conductor

# Capacitors

Let's now solve for the electric field in some conductor problems. The simplest examples are capacitors. These are a pair of conductors, one carrying surface charge density $\sigma$, the other $-\sigma$.

**Assumptions:**
Flat, parallel surfaces;
distance d between plates is much smaller than $\sqrt{A}$ where A is the area of the plates




$\sigma = Q/A$

$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{z}$

Conclusion:

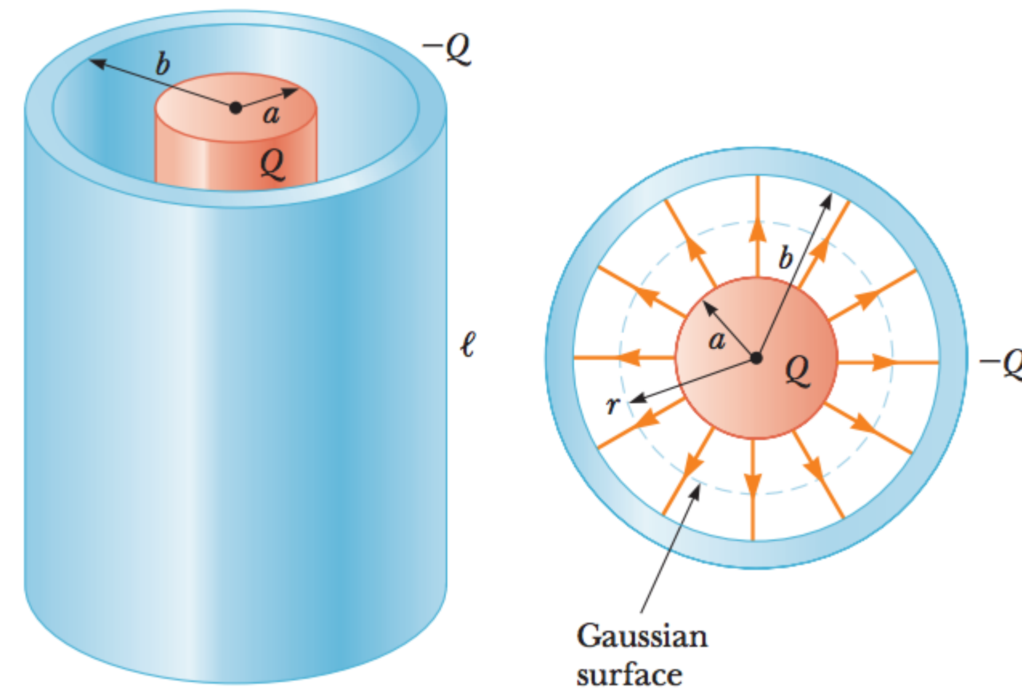
The field is $\frac{\sigma}{\epsilon_0}$ between the plates and points to the right; elsewhere it is zero.

We define capacitance as $C = \frac{Q}{V}$, where $V$ is the voltage or potential difference between plates.

Since $E = -dV/dz$ is constant, we have $V = -Ez + const. \implies V = V(0) - V(d) = Ed = \frac{Qd}{A\epsilon_0} \implies C = \frac{A\epsilon_0}{d}$.

**A solid cylindrical conductor of radius *a* and charge *Q* is coaxial with a cylindrical shell of negligible thickness, radius *b>a*, and charge -*Q*. Find the capacitance of this cylindrical capacitor if its length is L.**

Due to the cylindrical symmetry of the system, we choose our Gaussian surface to be a coaxial cylinder with length *l*< *L* and radius *r* where *a<r<b.*




$$\oint_S \vec{E}.d\vec{a} = EA = E2\pi r l = \frac{\lambda l}{\epsilon_0} \implies E = \frac{\lambda}{2\pi\epsilon_0 r}$$

Notice that the electric field is non-vanishing only in the region *a<r<b* . For *r < a* , the enclosed charge is *zero* since any net charge in a conductor must reside on its surface. Similarly, the enclosed charge is zero for *r > b,* since the Gaussian surface encloses equal but opposite charges from both conductors.

$$\Delta V = V_b - V_a = -\int_a^b E_s ds = -\frac{\lambda}{2\pi\epsilon_0}\int_a^b \frac{dr}{r} = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$$

**Capacitance crucially depends on the geometric factors *L*,*a* and *b***

$$C = \frac{Q}{|\Delta V|} = \frac{\lambda L}{\lambda \ln(b/a)/2\pi\epsilon_0} = \frac{2\pi\epsilon_0 L}{\ln(b/a)}$$

## Work done to charge a capacitor :

$q$= charge of the positive plate

Potential difference = $q/C$

To charge by a small amount $dq$ one has to work $=\dfrac{q}{C}dq$

Total work necessary to charge by amount Q is

$$W = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

Where $Q = CV$