3.05. Prove the identities:
(a)

$$\delta\Big(y(x)\Big) = \sum_i \frac{\delta(x - x_i)}{|\frac{dy}{dx}|_{x=x_i}} ,$$

where x_i's are roots of the equation $y(x) = 0$.
(b)

$$\frac{d\theta}{dx} = \delta(x) ,$$

where $\theta(x)$ be the step function, defined as

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{for } x \leq 0 . \end{cases}$$

(a) Consider the integration:

$$\int_{-\infty}^{+\infty} dx\ f(x)\ \delta\Big(y(x)\Big)$$

where $f(x)$ is a well behaved arbitrary function within $-\infty$ to $+\infty$.

Remember that $y(x) = 0$ at $x = x_i$ where $i = 1, 2, 3, \ldots, s$ (say)

i.e. $y(x) = 0$ has s no. of roots

Now since $\delta(y(x))$ is non-vanishing only at $y(x)=0$; it is sufficient to expand $y(x)$ around $x = x_i$ for our purpose.

i.e. $y(x) \simeq \cancel{y(x_i)}^{\,0} + (x - x_i)\left(\frac{dy}{dx}\right)_{x=x_i}$

$$= (x - x_i)\left(\frac{dy}{dx}\right)_{x=x_i}$$

Note that we kept upto first order in the expansion. It can be checked that this is enough for our main perpose and neglecting higher order terms will not make any loss of generality.

Under this, the integration can be subdevided into intervals around $x = x_i$'s ; i.e. where y vanishes.

$$\Rightarrow \int_{-\infty}^{+\infty} f(x)\, \delta\big(y(x)\big)\, dx$$

$$= \sum_{i=1}^{S} \int_{x_i - \epsilon}^{x_i + \epsilon} f(x)\, \delta\left((x - x_i) \left(\frac{dy}{dx}\right)_{x = x_i} \right) dx$$

small quantity and $\epsilon > 0$ **

Since $y(x)$ vanishes at each $x = x_i$; we have $\delta(y(x)) = \infty$ at each $x = x_i$. $\Rightarrow$ total area under the curve is sum of all individual curves.

** Since $\int_{-\infty}^{+\infty} dx\, f(x)\, \delta(x)$ is same as $\int_{0-\epsilon}^{0+\epsilon} dx\, f(x)\, \delta(x)\, dx$.

First consider $\left(\frac{dy}{dx}\right)_{x=x_i}$ is +ve.

Put $(x-x_i)\left(\frac{dy}{dx}\right)_{x=x_i} = z$

$$\therefore \quad dx = \frac{dz}{\left(\frac{dy}{dx}\right)_{x=x_i}}$$

Then $\displaystyle\int_{-\infty}^{+\infty} f(x)\,\delta\big(y(x)\big)\,dx$

$$= \sum_{i=1}^{S} \int_{-\epsilon}^{+\epsilon} f\left(\frac{z}{\left(\frac{dy}{dx}\right)_{x=x_i}} + x_i\right) \delta(z)\, \frac{dz}{\left(\frac{dy}{dx}\right)_{x=x_i}}$$

$$= \sum_{i=1}^{S} \frac{f(x_i)}{\left(\frac{dy}{dx}\right)_{x=x_i}}$$

Similarly, for $\left(\frac{dy}{dx}\right)_{x=x_i}$ is -ve:

$$\int_{-\infty}^{+\infty} f(x)\,\delta\big(y(x)\big)\,dx = \sum_{i=1}^{S} \frac{f(x_i)}{-\left(\frac{dy}{dx}\right)_{x=x_i}}$$

where one has to use the fact that $\delta(x)$ is an even function; i.e. $\delta(-x) = \delta(x)$.

$\therefore$ For both +ve and -ve values of $\left(\frac{dy}{dx}\right)_{x=x_i}$, one can write:

$$\int_{-\infty}^{+\infty} f(x)\,\delta(y(x))\,dx = \sum_{i=1}^{S} \frac{f(x_i)}{\left|\frac{dy}{dx}\right|_{x=x_i}} ;$$

where we have used the fact that

$$\left|\frac{dy}{dx}\right|_{x=x_i} = \begin{cases} +\left(\frac{dy}{dx}\right)_{x=x_i} & \text{for } \left(\frac{dy}{dx}\right)_{x=x_i} > 0 \\ -\left(\frac{dy}{dx}\right)_{x=x_i} & \text{for } \left(\frac{dy}{dx}\right)_{x=x_i} < 0. \end{cases}$$

The above can further be written as

$$\int_{-\infty}^{+\infty} f(x)\,\delta(y(x))\,dx = \sum_{i=1}^{S} \frac{f(x_i)}{\left|\frac{dy}{dx}\right|_{x=x_i}}$$

$$\equiv \sum_{i=1}^{S} \int_{-\infty}^{+\infty} f(x) \frac{\delta(x-x_i)}{\left|\frac{dy}{dx}\right|_{x=x_i}}\,dx$$

Now since $f(x)$ is chosen completely arbitrarily, the above yields:

$$\delta(y(x)) = \sum_{i=1}^{S} \frac{\delta(x-x_i)}{\left|\frac{dy}{dx}\right|_{x=x_i}}$$

(b) Consider an arbitrary function $f(x)$ which is well behaved with interval $-\infty$ to $+\infty$.

Let us now start with the integration

$$\int_{-\infty}^{+\infty} dx\, f(x)\, \frac{d\theta}{dx}$$

integration by parts

$$= \underbrace{f(x)\,\theta(x)\Big|_{-\infty}^{+\infty}} - \int_{-\infty}^{+\infty} dx\, \frac{df(x)}{dx}\,\theta(x)$$

in the lower limit $\theta(x) = 0$ and hence this term vanishes for $x \to -\infty$.

$$= \underbrace{f(\infty)} - \int_{-\infty}^{+\infty} dx\, \frac{df}{dx}\,\theta(x)$$

as for $x > 0$, $\theta(x) = 1$

$$= f(\infty) - \int_{0}^{\infty} dx\, \frac{df}{dx}$$

$$= f(\infty) - \Big[f(x) \Big]_0^{\infty}$$

$$= f(\infty) - f(\infty) + f(0)$$

$$= f(0) \equiv \int_{-\infty}^{+\infty} \delta(x) f(x) \, dx .$$

Now as $f(x)$ is completely arbitrary, one obtains:

$$\frac{d\theta}{dx} = \delta(x)$$