Tut. 5

5.01. (a) Two spherical conducting shells of radii r_a and r_b are arranged concentrically and are charged to the potentials V_a and V_b, respectively. If $r_b > r_b$, find the potential at points between the shell, and at points $r > r_b$.

(b) Two long cylindrical shells of radius r_a and r_b arranged coaxially and are charged to the potentials V_a and V_b, respectively. Find the potential at points between the cylindrical shells.

Here we have to solve the Laplace's equation : $\nabla^2 V = 0$ as there is no charges in the regions of interest ; under some specific boundary conditions (BC).

(a) Express Laplace's eqn in Spherical polar coordinates :

<u>Note:</u> Due to the symmetry, V will be independent of θ and φ ; and is function of r only.

$$\Rightarrow \nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0$$

or, $$r^2 \frac{dV}{dr} = c \quad \text{(Constant)}$$

or, $$dV = c\,\frac{dr}{r^2}$$

$$\Rightarrow \boxed{V = -\frac{c}{r} + k}$$ another constant.

These constants will be fixed by BC.

⊛ For region between the shells, we have two boundaries:

$$\left.\begin{aligned} V &= V_a \quad \text{at} \quad r = r_a \\ \text{and} \quad V &= V_b \quad \text{at} \quad r = r_b \end{aligned}\right\}$$

$$\therefore \quad V_a = -\frac{c}{r_a} + k \quad \text{and} \quad V_b = -\frac{c}{r_b} + k$$

Solving these two : $c = \left(\dfrac{V_a - V_b}{r_a - r_b}\right) r_a r_b$

and $k = \dfrac{V_a r_a - V_b r_b}{r_a - r_b}$.

$$\therefore \boxed{V = -\left(\frac{V_a - V_b}{r_a - r_b}\right)\frac{r_a r_b}{r} + \frac{V_a r_a - V_b r_b}{r_a - r_b}}.$$

⊛ For the region $r > r_b$, we have two boundaries:

$V \to 0$ at $r \to \infty$

$V = V_b$ at $r = r_b$.

First BC $\Rightarrow$ $k = 0$

$\therefore V = -C/r$

2^{nd} BC: $V_b = -\dfrac{C}{r_b}$ $\Rightarrow$ $C = -V_b r_b$.

$$\therefore \boxed{V = \frac{V_b r_b}{r}}$$

(b) Express the Laplace's eqn in Cylindrical coordinates:

Due to symmetry, V will be independent of ϕ, z.

Then $\nabla^2 V = \dfrac{1}{s}\dfrac{d}{ds}\left(s\dfrac{dV}{ds}\right) = 0$

$\Rightarrow s\dfrac{dV}{ds} = c$ (constant)

or, $c\dfrac{ds}{s} = dV$

To make dimensionless, introduce a length scale, say l.

Then $c\dfrac{ds/l}{s/l} = dV$

Put $s/l = x$

$$\Rightarrow \quad c \frac{dx}{x} = dV$$

or $\quad V = c \ln x + k$ → another constant.

$$\therefore \quad V(s) = c \ln\left(\frac{s}{l}\right) + k$$

BC: $\left.\begin{aligned} V(s = r_a) &= V_a \\ \text{and } V(s = r_b) &= V_b \end{aligned}\right\}$

$$\therefore \quad V_a = c \ln\left(\frac{r_a}{l}\right) + k \quad \text{and} \quad V_b = c \ln\left(\frac{r_b}{l}\right) + k$$

Solving these: $\quad c = \dfrac{V_a - V_b}{\ln(r_a/r_b)}$

and $\quad k = \dfrac{V_b \ln(r_a/l) - V_a \ln(r_b/l)}{\ln(r_a/r_b)}$

$$\therefore \boxed{V = \frac{V_a - V_b}{\ln(r_a/r_b)} \ln\left(\frac{s}{l}\right) + \frac{V_b \ln\left(\frac{r_a}{l}\right) - V_a \ln\left(r_b/l\right)}{\ln(r_a/r_b)}}$$

This length scale "l" can be associated with the arbitraryness of potential V i.e. V is defined upto some additive constant.

5.02. Consider a grounded conducting sphere of radius R. A charge q is placed at a distance $a > R$ on the z-axis. The image charge $q' = -Rq/a$ is kept at distance $b = R^2/a$ on the z-axis.

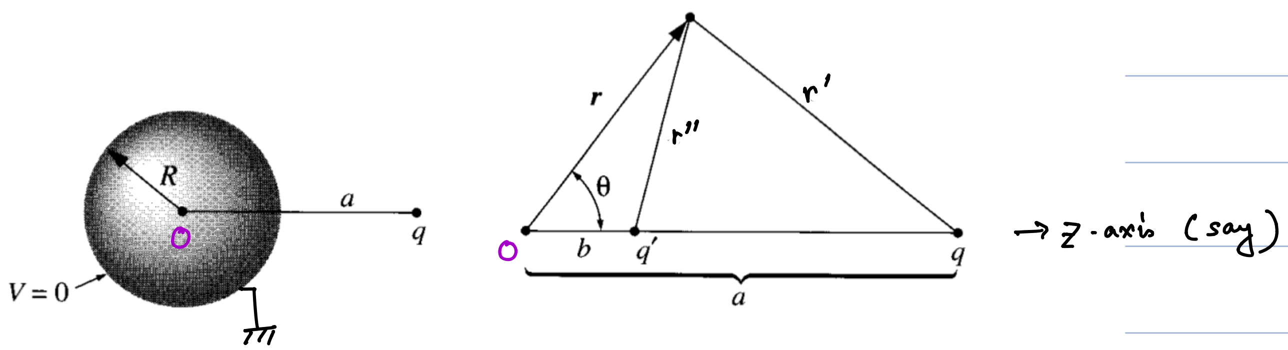
(a) Using the law of cosines, show that $V(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r'} + \frac{q'}{r''}\right)$ (where r' and r'' are the distances from q and q', respectively) can be written as follows:

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{1}{\sqrt{R^2+(ar/R)^2-2ar\cos\theta}}\right]$$

where r and θ are the usual spherical polar coordinates. In this form it is obvious that $V = 0$ on the sphere $r = R$.

(b) Find the induced charge on the sphere, as a function of θ. Integrate this to get the total induced charge.

(c) Calculate the energy of this configuration.



(a) $$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r'} + \frac{q'}{r''}\right)$$

Now,

$$r' = \left(r^2 + a^2 - 2ra\cos\theta\right)^{1/2}$$

and $$r'' = \left(r^2 + b^2 - 2rb\cos\theta\right)^{1/2} = \left(r^2 + \frac{R^4}{a^2} - \frac{2rR^2}{a}\cos\theta\right)^{1/2}$$

$$\therefore \frac{q'}{r''} = -\frac{Rq}{a}\frac{1}{\left(r^2 + \frac{R^4}{a^2} - \frac{2rR^2}{a}\cos\theta\right)^{1/2}}$$

$$= -\frac{Rq}{a}\frac{1}{\frac{R}{a}\left(R^2 + \left(\frac{ar}{R}\right)^2 - 2ra\cos\theta\right)^{1/2}}$$

$$= -\frac{q}{\left(R^2 + \left(\frac{ar}{R}\right)^2 - 2ra\cos\theta\right)^{1/2}}$$

$$\therefore \quad V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{1}{\sqrt{R^2+\left(\frac{ar}{R}\right)^2-2ra\cos\theta}}\right]$$

Note: $V = 0$ at $r = R$.

(b) Induced surface charge density is given by

$$\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial n}\right|_{r=R} = -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R}.$$

From the above expression for V, we find:

$$\left.\frac{\partial V}{\partial r}\right|_{r=R} = \frac{q}{4\pi\epsilon_0}\Bigg[-\frac{1}{2}\left(r^2+a^2-2ra\cos\theta\right)^{-3/2}(2r-2a\cos\theta)$$
$$\left.+\frac{1}{2}\left(R^2+\left(\frac{ar}{R}\right)^2-2ra\cos\theta\right)^{-3/2}\left(\frac{2ra^2}{R^2}-2a\cos\theta\right)\right]_{r=R}$$

$$= \frac{q}{4\pi\epsilon_0}\Big[-\left(R^2+a^2-2Ra\cos\theta\right)^{-3/2}(R-a\cos\theta)$$
$$+\left(R^2+a^2-2Ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R}-a\cos\theta\right)\Big]$$

$$= \frac{q}{4\pi\epsilon_0}\left(R^2+a^2-2Ra\cos\theta\right)^{-3/2}\left[\frac{a^2}{R}-a\cos\theta-R+a\cos\theta\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left(R^2+a^2-2Ra\cos\theta\right)^{-3/2}\left(\frac{a^2-R^2}{R}\right)$$

$$= \frac{q}{4\pi\epsilon_0 R}\left(a^2-R^2\right)\left(R^2+a^2-2Ra\cos\theta\right)^{-3/2}.$$

$$\therefore \; \sigma(\theta) = -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R}$$

$$= \frac{q}{4\pi R}\left(R^2 - a^2\right)\left(R^2 + a^2 - 2Ra\cos\theta\right)^{-3/2}$$

(Induced surface charge density).

Total induced charge :

$$q_{induced} = \int \sigma(\theta) \underbrace{da}_{R^2 \sin\theta \, d\theta \, d\phi}$$

$$= \frac{q}{4\pi R}\left(R^2 - a^2\right) R^2 \int_{\theta=0}^{\pi} \left(R^2 + a^2 - 2Ra\cos\theta\right)^{-3/2} \sin\theta \, d\theta \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{= 2\pi}$$

$$= \frac{qR}{2}\left(R^2 - a^2\right) \int_{\theta=0}^{\pi} d\theta \sin\theta \left(R^2 + a^2 - 2Ra\cos\theta\right)^{-3/2}$$

$$= \frac{qR}{2}\left(R^2 - a^2\right)\left(\frac{1}{Ra}\right) \int_{|R-a|}^{R+a} dy \, y \; y^{-3}$$

Put $R^2 + a^2 - 2Ra\cos\theta = y^2$

$\therefore \; 2Ra\sin\theta \, d\theta = 2y \, dy$

or, $\sin\theta \, d\theta = \frac{y}{Ra} dy$

$$= -\frac{q}{2a}\left(R^2 - a^2\right)\left[\frac{1}{y}\right]_{|R-a|}^{R+a}$$

Since we are interested on the surface of the sphere; here $R < a \Rightarrow |R-a| = a - R$.

$$\therefore \; q_{induced} = -\frac{q}{2a}(R^2-a^2)\left(\frac{1}{R+a} - \frac{1}{a-R}\right)$$

$$= -\frac{q}{2a}(R^2-a^2)\;\frac{a-R-R-a}{a^2-R^2}$$

$$= \frac{q}{2a}(-2R) = -\frac{qR}{a}$$

<u>Note:</u> Here $q_{induced}$ is q'.

$\Rightarrow$ q' can be considered as a image charge due to q.

(c) Energy of this configuration is determined by

$$W = -\underbrace{\int_{\infty}^{a} \vec{F}\cdot d\vec{r}}_{\text{work done to bring } q \text{ charge from infinity to } a.}$$

Here $\vec{F}$ is the force exerted on charge q by the charge induced on the sphere.

Since, induced charge is equivalent to the image charge q', the force is given by the force between charges q and q'. With q placed at $z\hat{z}$, the force is

$$\vec{F} = \frac{qq'}{4\pi\epsilon_0(z-b)^2}\,\hat{z}$$

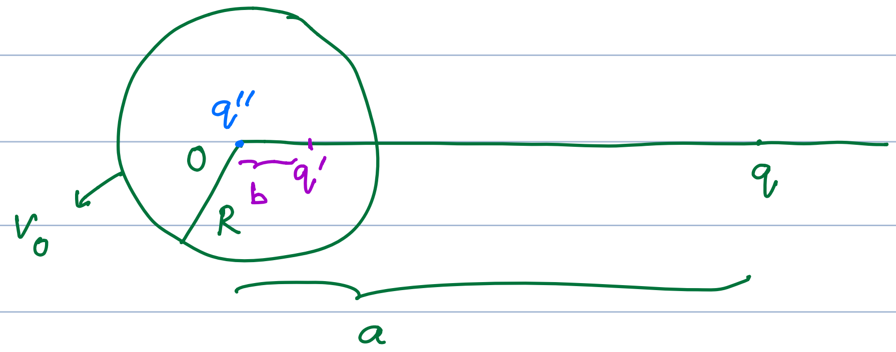
$$= \frac{q}{4\pi\epsilon_0\left(z-\frac{R^2}{z}\right)^2}\left(-\frac{Rq}{z}\right)\hat{z} = -\frac{1}{4\pi\epsilon_0}\frac{q^2Rz}{(z^2-R^2)^2}\,\hat{z}$$

where, $b = R^2/z$ and $q' = -\frac{Rq}{z}$.

$$\therefore\ W = \frac{1}{4\pi\epsilon_0}\left(q^2 R\right)\int_{\infty}^{a} \frac{z\,dz}{\left(z^2 - R^2\right)^2}$$

$$= -\frac{1}{4\pi\epsilon_0}\,\frac{q^2 R}{2\left(a^2 - R^2\right)}$$

5.03. Consider a point charge q situated at a distance a from the center of a grounded conducting sphere of radius R. $(a > R)$ The same basic model will handle the case of a sphere at any potential V_0 (relative to infinity) with the addition of a second image charge. What charge should you use, and where should you put it? Find the force of attraction between a point charge q and a neutral conducting sphere.



This situation can be handled in the following way.

Suppose initially the sphere is at zero potential.

Then, by the previous problem, the image charge due to q is $q' = -\frac{qR}{a}$ at $b = \frac{R^2}{a}$ (as shown in figure).

Now to incorporate the real fact that the sphere is at potential V_0, we can put a q'' charge at O (center of the sphere); s.t.

$$V_0 = \frac{q''}{4\pi\epsilon_0 R}.$$

This is fine since the charge q'' at center O makes the surface equipotential.

$$\therefore \quad q'' = 4\pi\epsilon_0 R V_0.$$

$\Rightarrow$ Now, we have the total induced charge as $q' + q''$ on the surface of the sphere.

The total force on q is calculated by evaluating the force due to q and q' as well as q''.

$$\therefore \quad F = \frac{q q'}{4\pi\epsilon_0 (a-b)^2} + \frac{q q''}{4\pi\epsilon_0 a^2}$$

where $b = \frac{R^2}{a}$

$q' = -\frac{qR}{a}$.

Now for neutral sphere : $q' + q'' = 0$

$\Rightarrow q'' = -q' = \frac{qR}{a}$.

$$\therefore \quad F = \frac{q q'}{4\pi\epsilon_0} \left[\frac{1}{(a-b)^2} - \frac{1}{a^2} \right]$$

$$= -\frac{q^2 R}{4\pi\epsilon_0 a} \left[\frac{1}{\left(a - \frac{R^2}{a}\right)^2} - \frac{1}{a^2} \right]$$

$$= -\frac{q^2 R}{4\pi\epsilon_0 a} \underbrace{\frac{a^4 - (a^2 - R^2)^2}{a^2 (a^2 - R^2)^2}}$$

$$\frac{-R^4 + 2a^2R^2}{a^2(a^2-R^2)^2} = \frac{R^2(2a^2 - R^2)}{a^2(a^2-R^2)^2}$$

$$= -\frac{q^2}{4\pi\epsilon_0} \left(\frac{R}{a}\right)^3 \frac{2a^2 - R^2}{(a^2-R^2)^2}.$$

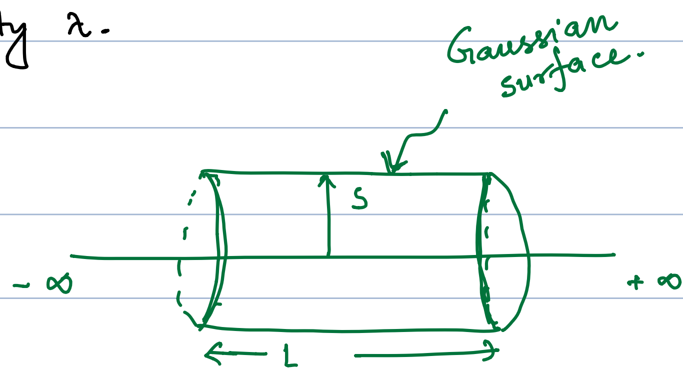
Note: Since $a > R$, here $F < 0 \Rightarrow$ Force is attractive.

5.04. Two long, straight copper pipes, each of radius R, are held with their axes at a distance $2d$ apart. One is at potential V_0, the other at $-V_0$. Find the potential everywhere.

First of all let us calculate the field due to a uniformly charged infinite line of charge of linear density λ.

Using Gauss's law:

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{encl.}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$$



$$\Rightarrow E \cdot 2\pi s L = \frac{\lambda L}{\epsilon_0}$$

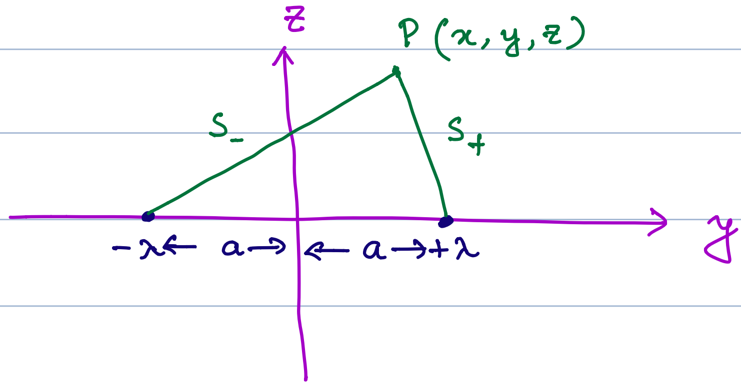
or, $$\boxed{\vec{E}(s) = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}}$$

Now we calculate the potential at any distance s. In this case, we can not set the reference point at ∞ as the charge itself extends to ∞. Let us then set the reference point at $s = e$ (say).

Then $$V(s) = -\int_e^s \vec{E} \cdot d\vec{l} = -\frac{\lambda}{2\pi\epsilon_0}\int_e^s \frac{dl}{l}$$

$$\therefore \boxed{V(s) = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s}{e}\right)}$$

Now in the given problem, let us replace the copper pipes (Oriented parallel to the z-axis as shown) by infinite lines of image charge with linear densities $\pm\lambda$, respectively, at $y = \pm a$.



Then the total potential at any point $P(x,y,z)$ is

$$V_P = -\frac{(-\lambda)}{2\pi\epsilon_0} \ln\left(\frac{S_-}{e}\right) - \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{S_+}{e}\right)$$

$$= -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{S_+}{S_-}\right) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{S_-}{S_+}\right)$$

$$\boxed{V_P = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{S_-}{S_+}\right)}$$

where

$$S_\pm = \sqrt{(y \mp a)^2 + z^2}$$

$$\therefore V_P = \frac{\lambda}{2\pi\epsilon_0} \ln\left[\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right]^{1/2} = \frac{\lambda}{4\pi\epsilon_0} \ln\left[\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right]$$

The above equation yields the equations of the family of equipotential surfaces, given by,

$$\frac{4\pi\epsilon_0 V_P}{\lambda} = \ln\left[\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right] = \text{Constant}, \quad \forall \text{ const.} \in \mathbb{R}.$$

or, $$\frac{(y+a)^2+z^2}{(y-a)^2+z^2} = \exp\left(\frac{4\pi\epsilon_0 V_P}{\lambda}\right) = \ln(\text{Const.})$$

Since the surfaces of the conducting wires are equipotentials, then for right conductor, $V_P = V_0$.

$$\Rightarrow \frac{(y+a)^2+z^2}{(y-a)^2+z^2} = \exp\left(\frac{4\pi\epsilon_0 V_0}{\lambda}\right) = k \text{ (say)} \in \mathbb{R}.$$

$$\Rightarrow (y+a)^2+z^2 = k\left[(y-a)^2+z^2\right]$$

or, $$\boxed{y^2+z^2+a^2-2ay\left(\frac{k+1}{k-1}\right) = 0}$$

which represents an equation of circle on the y-z plane.

Now, the general equation of a circle with center $(y_0,0)$ and radius R on the y-z plane is

$$(y-y_0)^2 + z^2 = R^2$$

or, $$\boxed{y^2+z^2+(y_0^2-R^2)-2yy_0 = 0.}$$

Comparing, we conclude that the family of equipotentials are circles (as shown below) and in particular the axis of the right copper wire is located at

$$y_0 = a\frac{k+1}{k-1} = d \quad \text{(as given in the problem)}$$

Similar will be also for left conductor.

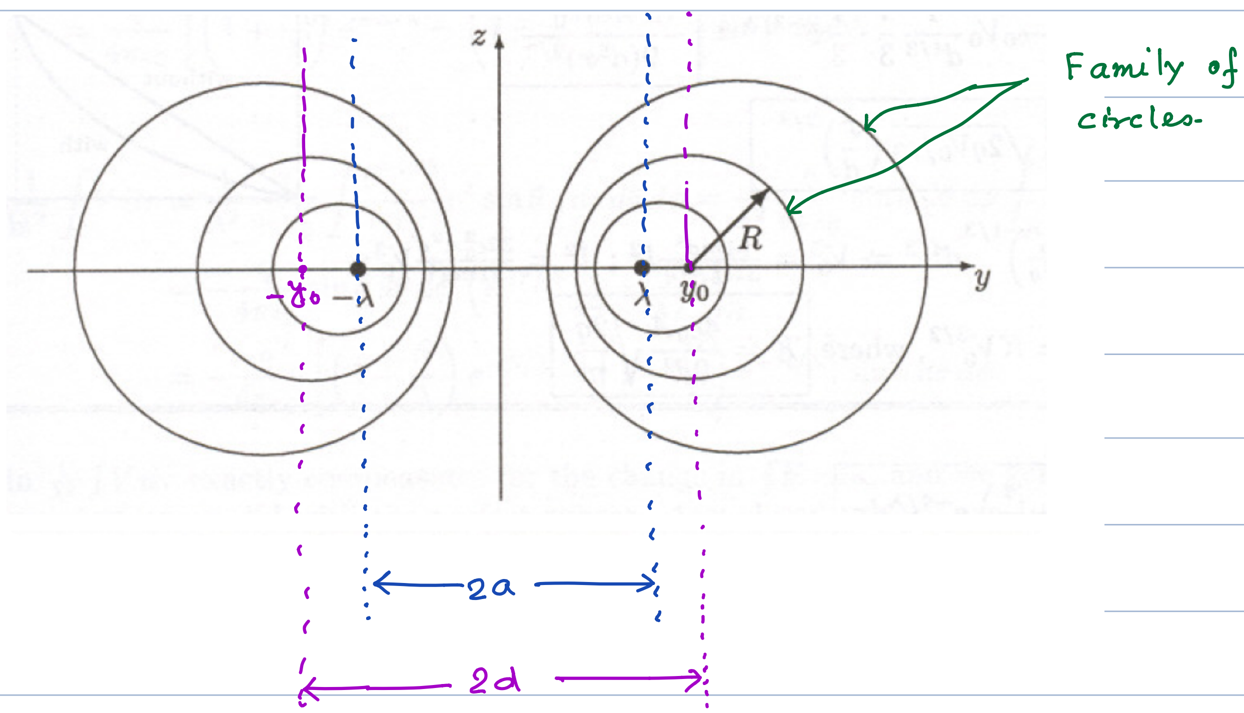
In this case $V_p = -V_0$

$$\therefore\ y_0 = a\left(\frac{\frac{1}{k}+1}{\frac{1}{k}-1}\right) = a\left(\frac{1+k}{1-k}\right) = -d$$

(as given in the problem).

$$\therefore\ d = a\left(\frac{k+1}{k-1}\right)$$

So for both the cases $\boxed{d = \frac{a(k+1)}{k-1}}$



Also, we have $a^2 = y_0^2 - R^2 = d^2 - R^2$

$$\Rightarrow a = \sqrt{d^2 - R^2}.$$

Now,

$$d = \frac{a(k+1)}{k-1} = a\left(\frac{e^{\frac{4\pi\epsilon_0 V_0}{\lambda}} + 1}{e^{\frac{4\pi\epsilon_0 V_0}{\lambda}} - 1}\right)$$

$$= a\left[\frac{e^{\frac{2\pi\epsilon_0 V_0}{\lambda}} + e^{-\frac{2\pi\epsilon_0 V_0}{\lambda}}}{e^{\frac{2\pi\epsilon_0 V_0}{\lambda}} - e^{-\frac{2\pi\epsilon_0 V_0}{\lambda}}}\right]$$

$$\therefore \boxed{d = a \coth\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)}$$

Again, $a^2 = y_0^2 - R^2 = a^2\left(\frac{k+1}{k-1}\right)^2 - R^2$

or, $R^2 = \left[\left(\frac{k+1}{k-1}\right)^2 - 1\right] a^2 = \frac{4k}{(k-1)^2} a^2$

or, $R = a \frac{2\sqrt{k}}{k-1}$

$$= 2a \frac{e^{\frac{2\pi\epsilon_0 V_0}{\lambda}}}{e^{\frac{4\pi\epsilon_0 V_0}{\lambda}} - 1}$$

$$= \frac{2a}{e^{\frac{2\pi\epsilon_0 V_0}{\lambda}} - e^{-\frac{2\pi\epsilon_0 V_0}{\lambda}}} = \frac{a}{\sinh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)}$$

$$\Rightarrow \quad \boxed{a = R \sinh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)}$$

$$\therefore \quad \frac{d}{R} = \sinh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right) \times \coth\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)$$

$$\Rightarrow \quad \boxed{\frac{d}{R} = \cosh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)} \quad \Rightarrow \quad \boxed{\lambda = \frac{2\pi\epsilon_0 V_0}{\cosh^{-1}\left(\frac{d}{R}\right)}}$$

$\therefore$ The required expression for potential is

$$V_P = \frac{\lambda}{4\pi\epsilon_0} \ln\left[\frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}\right]$$

$$= \frac{V_0}{2\cosh^{-1}\left(\frac{d}{R}\right)} \ln\left[\frac{\left(y + \sqrt{d^2-R^2}\right)^2 + z^2}{\left(y - \sqrt{d^2-R^2}\right)^2 + z^2}\right]$$

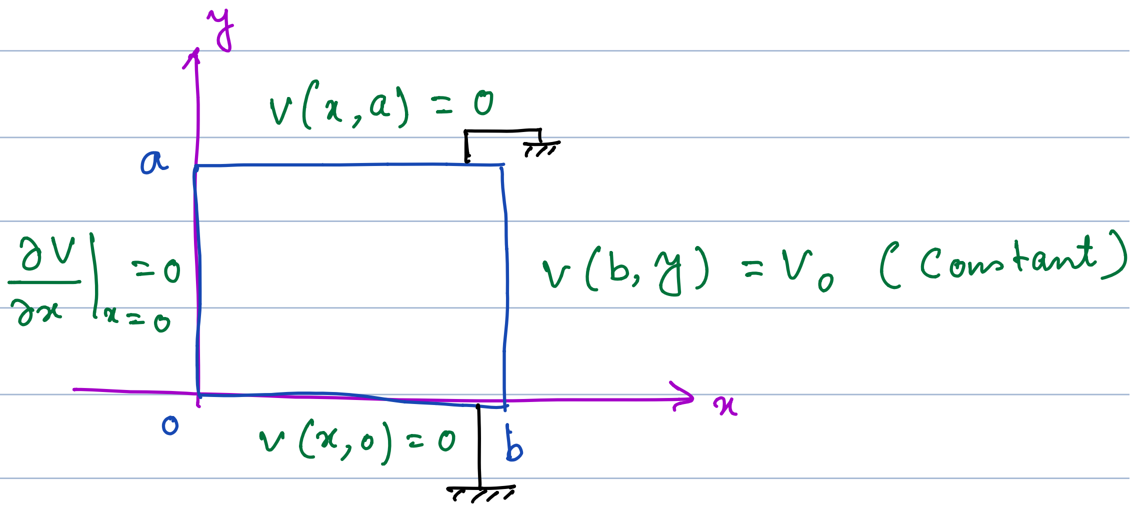
$$\forall \, (y, z) \in \mathbb{R}^2.$$

5.05. A rectangular pipe, running parallel to the z-axis (from $-\infty$ to ∞), has two grounded metal sides, at $y = 0$ and at $y = a$. At $x = 0$ side, the normal component of the electric field is zero, that is $\partial V/\partial x = 0$, where V is the potential function. The fourth side at $x = b$ is maintained at a constant potential V_0.

(a) Use the method of variable separation and write down the product solutions which satisfy boundary conditions at $y = 0, y = a$ and $x = 0$.

(b) Find the potential everywhere inside the pipe. Leave your answer in series form.

(c) What is the induced charge density on the $y = a$ surface? Again leave your answer in series form.



Since the rectangular pipe is running from $z = -\infty$ to $z = +\infty$; the potential inside the pipe is independent of z;

i.e. $V = V(x,y)$

$\Rightarrow$ Laplace's equation : $\nabla^2 V = 0$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

BCs:

$$\left.\begin{array}{ll} (i) & V(x,0) = 0 \\ (ii) & V(x,a) = 0 \\ (iii) & \left.\frac{\partial V}{\partial x}\right|_{x=0} = 0 \\ (iv) & V(b,y) = V_0, \text{ constant.} \end{array}\right\}$$

(a) Use separation of variables technique:

$$V(x,y) = X(x)\,Y(y)$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = k^2 \text{ (const.)}$$

$$\left.\begin{aligned} \therefore X &= A e^{kx} + B e^{-kx} \\ \text{and } Y &= C \sin(ky) + D \cos(ky) \end{aligned}\right\}$$

BC (i) $\Rightarrow$ $D = 0$

$$\therefore V(x,y) = \left(A e^{kx} + B e^{-kx}\right) C \sin(ky)$$

$$= \left(A e^{kx} + B e^{-kx}\right) \sin ky$$

(Absorbing C in A and B)

Now, BC (ii) $\Rightarrow$ $\sin ka = 0$

$\Rightarrow ka = n\pi$ with $n = 1, 2, 3, \ldots$

[Here n can not be zero, otherwise V trivially vanishes. Negavitive integer values of n do not give independent solutions.].

$$\Rightarrow \boxed{k = \frac{n\pi}{a}.} \quad \text{with } n = 1, 2, 3, \ldots$$

BC (iii) $\Rightarrow$ $Ak - Bk = 0$

$\Rightarrow A = B$. (As $k \neq 0$)

$\therefore$ Thus,

$$V(x, y) = A\left(e^{kx} + e^{-kx}\right) \sin(ky)$$

$$= 2A \cosh(kx) \sin(ky)$$

$\therefore$ The eigen solutions are: (factor of 2 is absorbed in A)

$$V_n(x,y) = A_n \cosh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

where $n = 1, 2, 3, \ldots$

Full solution: $\boxed{V(x,y) = \sum_{n=1,2,3,\ldots} A_n \cosh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)}$

(b) Using BC (iv) $\Rightarrow$

$$V_0 = \sum_{n=1,2,3,\ldots} A_n \cosh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

To fix A_n:

Multiply both sides by $\sin\left(\frac{m\pi y}{a}\right)$ and then integrating over y for $y = 0$ to $y = a$, we obtain:

$$\int_0^a V_0 \sin\left(\frac{m\pi y}{a}\right) dy = \sum_{n=1,2,3,\ldots} A_n \cosh\left(\frac{n\pi b}{a}\right) \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \times dy$$

Now, $$\int_0^a dy\ \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) = \begin{cases} 0 & \text{for } m \neq n \\ \frac{a}{2} & \text{for } m = n. \end{cases}$$

$$\Rightarrow \int_0^a dy\ \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) = \frac{a}{2}\,\delta_{mn}.$$

$$\therefore V_0 \int_0^a \sin\left(\frac{m\pi y}{a}\right) dy = \sum_{n=1,2,3,\cdots} A_n \cosh\left(\frac{n\pi b}{a}\right) \times \frac{a}{2}\,\delta_{mn}$$

$$= \frac{a}{2} A_m \cosh\left(\frac{m\pi b}{a}\right)$$

$$\Rightarrow A_m = \frac{2}{a}\,\frac{V_0}{\cosh\left(\frac{m\pi b}{a}\right)} \int_0^a \sin\left(\frac{m\pi y}{a}\right) dy$$

$$= \frac{a}{m\pi}\left(1 - (-1)^m\right) = \begin{cases} 0 & \text{for } m \text{ even } (m = 2, 4, 6, \cdots) \\ \frac{2a}{m\pi} & \text{for } m \text{ odd } (m = 1, 3, 5, \cdots) \end{cases}$$

$$= \begin{cases} \dfrac{4V_0}{m\pi \cosh\left(\frac{m\pi b}{a}\right)} & \text{for } m \text{ odd.} \\ 0 & \text{for } m \text{ even.} \end{cases}$$

$$\therefore \boxed{V(x,y) = \sum_{n=1,3,5,\cdots} \frac{4V_0}{n\pi \cosh\left(\frac{n\pi b}{a}\right)} \cosh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)}$$

(c) Induced charge density is determined by

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}\bigg|_{\text{surface}}.$$

Here the surface is $y = a$.

$\therefore \hat{n} = -\hat{y}$ (As we are looking from the inside of the conductor)

$$\therefore \sigma = \epsilon_0 \frac{\partial V}{\partial y}\bigg|_{y=a}$$

$$= \epsilon_0 \sum_{n=1,3,5,\cdots} \frac{4V_0}{n\pi \cosh\left(\frac{n\pi b}{a}\right)} \cosh\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi y}{a}\right) \times \frac{n\pi}{a}\bigg|_{y=a}$$

$$= \sum_{n=1,3,5,\cdots} \frac{4\epsilon_0 V_0}{a \cosh\left(\frac{n\pi b}{a}\right)} \cosh\left(\frac{n\pi x}{a}\right) \cos(n\pi)$$

since allowed values of n are odd integers, the value of this term is (-1)

$$= -\sum_{n=1,3,5} \frac{4\epsilon_0 V_0}{a \cosh\left(\frac{n\pi b}{a}\right)} \cosh\left(\frac{n\pi x}{a}\right).$$