1.

(a) $\delta(\sin x)$ will be non-vanishing when $\sin x = 0$

i.e. $x = n\pi$; with $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

(b) Using the given formula we have

$$\int_{-\infty}^{+\infty} f(x)\, \delta(y(x))\, dx$$

$$= \sum_i \frac{f(x_i)}{\left|\frac{dy}{dx}\right|_{x=x_i}}$$

Here $f(x) = e^{-|x|}$ and $y(x) = \sin x$

with $x_i = i\pi$; $i = 0, \pm 1, \pm 2, \pm 3, \ldots$

$$\therefore \left|\frac{dy}{dx}\right|_{x=x_i=i\pi} = \left|\cos x\right|_{i\pi} = 1$$

$$\therefore \int_{-\infty}^{+\infty} dx\, e^{-|x|}\, \delta(\sin x)$$

$$= \sum_{i=0,\pm 1,\pm 2,\cdots} e^{-|i\pi|}$$

$$= 1 + \sum_{n=1}^{\infty} e^{-|in|} + \sum_{i=-1}^{-\infty} e^{-|in|}$$

$\nearrow$

(For $n=0$)

$$= 1 + \sum_{i=1}^{\infty} e^{-i\pi} + \sum_{i=-1}^{-\infty} e^{i\pi}$$

$$= 1 + \sum_{i=1}^{\infty} e^{-i\pi} + \sum_{i=1}^{\infty} e^{-i\pi} \quad \left[\text{Put } i \to -i \text{ in the last term.}\right]$$

$$= 1 + 2\sum_{i=1}^{\infty} e^{-i\pi}$$

$$= 1 + 2\left[e^{-\pi} + e^{-2\pi} + e^{-3\pi} + \cdots\right]$$

$$= 1 + \frac{2e^{-\pi}}{1 - e^{-\pi}} = \frac{1 + e^{-\pi}}{1 - e^{-\pi}}$$

$$= \frac{e^{\pi/2} + e^{-\pi/2}}{e^{\pi/2} - e^{-\pi/2}}$$

$$= \frac{\cosh \pi/2}{\sinh \pi/2} = \coth(\pi/2).$$

2.

(a) $\vec{D} = zs\cos^2\phi\,\hat{z}$

Use $\vec{\nabla}\cdot\vec{D} = \rho_f$ to find ρ_f.

Now, $\vec{\nabla}\cdot\vec{D} = \frac{\partial}{\partial z}\left(zs\cos^2\phi\right)$

$= s\cos^2\phi$

$\therefore \boxed{\rho_f = s\cos^2\phi}$

(b) Total volume charge (free)

$Q_f = \int \rho_f \, d\tau$

$= \int s\cos^2\phi \cdot s\,ds\,d\phi\,dz$

$= \int_{s=0}^{1} s^2\,ds \int_{\phi=0}^{2\pi} \cos^2\phi\,d\phi \int_{z=-2}^{+2} dz$

$= \left[\frac{s^3}{3}\right]_0^1 \times \pi \times \left[z\right]_{-2}^{+2}$

$= \frac{1}{3}\times\pi\times 4 = \frac{4\pi}{3}$ units.

3.

(a) Use Gauss' law:

$$\oint_S \vec{E} \cdot d\vec{s} = \frac{Q_{enclosed}}{\epsilon_0}$$

Here $Q_{enclosed} = \dfrac{Q}{4/3 \pi R^3} \times \dfrac{4}{3}\pi r^3$

$$= \frac{Q r^3}{R^3}.$$

Also,

$$\oint \vec{E} \cdot d\vec{s} = E \cdot 4\pi r^2$$

$$\therefore E = \frac{Q r^3}{\epsilon_0 R^3} \frac{1}{4\pi r^2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q r}{R^3}$$

Electric field is along radial direction.

$$\therefore \boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} \vec{r}}$$

(b) If ρ is the charge density, then the charge contained in $d\tau$ volume element at $\vec{r}$ is

$= \rho \, d\tau.$

where $\rho = \dfrac{Q}{4/3 \pi R^3}$

Force on that will be determined by the electric field at that point.

This is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} \vec{r}$$

[Evaluated in Part (a)]

$\therefore$ Force on this volume element is

$$\vec{f}\, d\tau = \rho\, d\tau\, \vec{E}$$

$$= \frac{Q}{\frac{4}{3}\pi R^3} d\tau \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} \vec{r}$$

$$= \left(\frac{3}{\epsilon_0}\right)\left(\frac{Q}{4\pi R^3}\right)^2 \vec{r}\, d\tau$$

$\therefore$ z - component of the force is given by

$$f_z\, d\tau = \vec{f}\, d\tau \cdot \hat{z}$$

$$= \left(\frac{3}{\epsilon_0}\right)\left(\frac{Q}{4\pi R^3}\right)^2 \vec{r} \cdot \hat{z}\, d\tau$$

$$= \left(\frac{3}{\epsilon_0}\right)\left(\frac{Q}{4\pi R^3}\right)^2 r \cos\theta \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi .$$

(c) Net force that the southern hemisphere exerts on northern hemisphere is

$$F_z = \int f_z \, d\tau$$

$$= \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \int_{r=0}^{R} r^3 \, dr \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \frac{R^4}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \times 2\pi$$

$$= \frac{3}{\epsilon_0} \frac{Q^2}{16\pi^2 R^6} \frac{R^4}{4} \times \frac{1}{2} \times 2\pi$$

$$= \frac{3}{\epsilon_0} \frac{Q^2}{64\pi R^2} = \frac{3Q^2}{64\pi\epsilon_0 R^2}.$$

4.

(a) Using Gauss' Law:

$$E \cdot 2\pi s l = \frac{\lambda l}{\epsilon_0}$$

where l is ~~Gauss~~ the length of Gaussian surface and s is the radius of it.

$$\therefore \vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$$

$$\therefore \text{Potential } V = -\int_{O}^{s} \vec{E}\cdot d\vec{l}$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\int_{O}^{s}\frac{ds}{s}$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s}{s_0}\right)$$

where s_0 is some length scale of the system and the reference pt O is chosen in such a way that $\frac{s}{s_0} = 1$ at ref. pt.

(b) (i) Boundary surfaces are at
$z = 0$ and $y^2+z^2 \gg d^2$

Boundary conditions:
- $V = 0$ at $z = 0$
- $V \to 0$ as $y^2+z^2 \gg d^2$

(ii) The potential in the region $z>0$ can be obtained by image method.

The image of induced charge on the plane can be ~~thought~~ taken as infinite line charge distribution ~~along~~ parallel to x-axis with density $(-\lambda)$. This must be at a distance $z=-d$.

$\therefore$ The potential at pt. (y,z) is

$$V = V_+ + V_-$$

where
$$V_+ = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_+}{s_0}\right)$$
$$V_- = -\frac{(-\lambda)}{2\pi\epsilon_0}\ln\left(\frac{s_-}{s_0}\right) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_-}{s_0}\right)$$

with
$$s_\pm = \sqrt{y^2+(z\mp d)^2}$$

<u>Note:</u> Since line charges are infinitely extended and parallel to x-axis, potential must be independent of x.

$$\therefore\ V = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_-}{s_+}\right)$$
$$= \frac{\lambda}{2\pi\epsilon_0}\ln\left[\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right]^{1/2}$$
$$= \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right]$$

(iii) Induced charged density on plane

$$\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}.$$

Here $$\left.\frac{\partial V}{\partial z}\right|_{z=0} = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{2(z+d)}{y^2+(z+d)^2} - \frac{2(z-d)}{y^2+(z-d)^2}\right]_{z=0}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\frac{2d}{y^2+z^2} + \frac{2d}{y^2+z^2}\right]$$

$$= \frac{\lambda}{4\pi\epsilon_0}\,\frac{4d}{y^2+z^2}$$

$$= \frac{\lambda d}{\pi\epsilon_0(y^2+z^2)}$$

$$\therefore \boxed{\sigma = -\frac{\lambda d}{\pi(y^2+z^2)}}$$

Question 5

Solⁿ: Given polarization is

$$\vec{P} = -\frac{\rho_0}{3}\vec{r}\left(1-\frac{3r}{4R}\right) \equiv P_r\,\hat{r}$$

(a) The bound volume density is given by

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right)$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left[-r^2\frac{\rho_0}{3}r\left(1-\frac{3r}{4R}\right)\right]$$

$$= \frac{1}{r^2}\frac{\rho_0}{3}\frac{\partial}{\partial r}\left[r^3-\frac{3r^4}{4R}\right]$$

$$= \frac{1}{r^2}\frac{\rho_0}{3}\left[3r^2-\frac{3r^3}{R}\right]$$

or, $$\boxed{\rho_b(r) = \rho_0\left(1-\frac{r}{R}\right)},$$

& the surface bound charge density is given by

$$\sigma_b = \left(\vec{P}\cdot\hat{n}\right)_s = \vec{P}\cdot\hat{r}\Big|_{r=R} = -\frac{\rho_0 r}{3}\left(1-\frac{3r}{4R}\right)\Big|_{r=R}$$

or, $$\boxed{\sigma_b = -\frac{\rho_0 R}{12}}$$

(b) Since the dielectric sphere is initially an uncharged one $Q_{free}=0$. Also it may is easy ~~best~~ to check that the total bound charge $Q_b = 0$. Hence, the electric field outside is zero as a consequence of Gauss' law.

For determining the field inside we may use Gauss law for the Electric Displacement

$$\oiint \vec{D}\cdot d\vec{S} = q_{f,encl} = 0 \Rightarrow \vec{D} = 0 \quad \forall\, r.$$

Using the constitutive relation $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$, we get

$$0 = \epsilon_0 \vec{E} + \vec{P}$$

or, $$\vec{E} = -\frac{1}{\epsilon_0}\vec{P} = +\frac{1}{\epsilon_0}\frac{\rho_0}{3}\vec{r}\left(1 - \frac{3r}{4R}\right)$$

or, $$\boxed{\vec{E}(\vec{r}) = \frac{\rho_0}{3\epsilon_0}\vec{r}\left(1 - \frac{3r}{4R}\right)}$$

(c) $$\frac{dE}{dr} = \frac{\rho_0}{3\epsilon_0}\left(1 - \frac{6r}{4R}\right)$$

and $$\frac{d^2E}{dr^2} = -\frac{2\rho_0}{4R\epsilon_0} = -\frac{\rho_0}{2R\epsilon_0} < 0 \quad \text{since } \rho_0 > 0 .$$

$\therefore$ $E = E_{max}$ at $r = r_{max}$

where, $$\left.\frac{dE}{dr}\right|_{r = r_{max}} = 0 = \frac{\rho_0}{3\epsilon_0}\left(1 - \frac{6r_{max}}{4R}\right)$$

$$\Rightarrow \boxed{r_{max} = \frac{2R}{3}}$$

and, $$E_{max} = \frac{\rho_0}{3\epsilon_0}\left[\frac{2R}{3} - \frac{3}{4R}\left(\frac{2R}{3}\right)^2\right]$$

$$= \frac{2\rho_0 R}{9\epsilon_0}\left[1 - \frac{\cancel{3}}{\underset{2}{\cancel{4R}}} \cdot \frac{\cancel{2R}}{\cancel{3}}\right]$$

or, $$\boxed{E_{max} = \frac{\rho_0 R}{9\epsilon_0}}.$$

6. Consider a vector field $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$, and the part of the paraboloid $z = \frac{x^2}{4} + \frac{y^2}{9}$ such that $z \leq 1$.

(a) Find the unit normal vector on the surface of the paraboloid corresponding to ~~the~~ traversing the contour of the paraboloid at $z = 1$ in the "positive" direction.

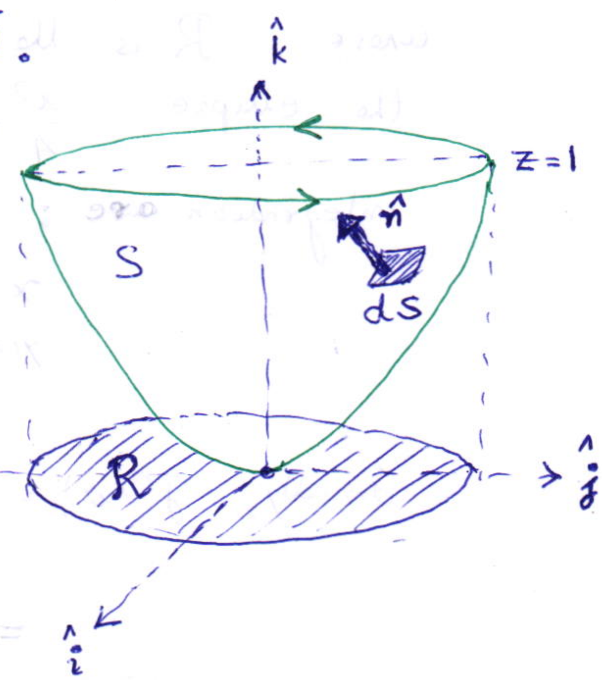
(b) Determine $\vec{\nabla} \times \vec{F}$.

(c) The "positive" flux through the open surface of the paraboloid may be expressed in the double integral form

$$\Phi = \iint_R g(x,y)\, dR,$$

where R is the projection of the paraboloid on the xy-plane. Determine the equation of the region R, the limits of integration in the above flux integral and the function $g(x,y)$.

(d) Evalute the flux Φ.



Sol^n:

(a) Given eqn. of parabolid

$$f(x,y,z) = z - \frac{x^2}{4} - \frac{y^2}{9} = 0$$

$\therefore$ Positive normal is

$$\hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{-\frac{x}{2}\hat{i} - \frac{2y}{9}\hat{j} + \hat{k}}{\sqrt{\frac{x^2}{4} + \frac{4y^2}{81} + 1}}$$

NOTE: This normal has a positive co-efficient for $\hat{k}$, hence the positive normal corresponding to the traversing around the contour at $z = 1$.

(b) $\vec{\nabla}\times\vec{F} = \hat{i}+\hat{j}+\hat{k}$

& $$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = \left(\frac{-\frac{x}{2}\hat{i}-\frac{2y}{9}\hat{j}+\hat{k}}{\sqrt{\frac{x^2}{4}+\frac{4y^2}{81}+1}}\right)\cdot(\hat{i}+\hat{j}+\hat{k})$$

$$= \frac{-\frac{x}{2}-\frac{2y}{9}+1}{\sqrt{\frac{x^2}{4}+\frac{4y^2}{81}+1}}$$

(c) $$\Phi = \iint_S \vec{F}\cdot\hat{n}\, dS = \iint_R (\vec{\nabla}\times\vec{F})\cdot\hat{n}\ \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \iint_R \frac{(\vec{\nabla}\times\vec{F})\cdot\hat{n}}{|\hat{n}\cdot\hat{k}|}\,dx\,dy$$

$$= \iint_R g(x,y)\ dR$$

where, R is the region on the xy-plane denoted by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$, and the limits of integration are :

$$x \rightarrow -2 \text{ to } +2$$
$$y(x) \rightarrow -3\sqrt{1-\frac{x^2}{4}} \text{ to } 3\sqrt{1-\frac{x^2}{4}}$$

Finally, $$g(x,y) = \frac{(\vec{\nabla}\times\vec{F})\cdot\hat{n}}{|\hat{n}\cdot\hat{k}|} = \frac{-\frac{x}{2}-\frac{2y}{9}+1}{\cancel{\sqrt{\frac{x^2}{4}+\frac{4y^2}{81}+1}}}\times\cancel{\sqrt{\frac{x^2}{4}+\frac{4y^2}{81}+1}}$$

$$= -\frac{x}{2}-\frac{2y}{9}+1$$

(d) $$\Phi = -\int_{-2}^{+2} dx \int_{-3\sqrt{1-\frac{x^2}{4}}}^{+3\sqrt{1-\frac{x^2}{4}}} dy \left(\cancel{\frac{x}{2}}+\cancel{\frac{2y}{9}}-1\right) = \int_{-2}^{+2} dx \int_{-3\sqrt{1-\frac{x^2}{4}}}^{3\sqrt{1-\frac{x^2}{4}}} dy = \underline{\underline{6\pi}}\,(\text{Area of ellipse})$$

odd integrals

Attached with the other file.