Physics II (PH 102)

Electromagnetism (Lecture 1)

Udit Raha (Saurabh Basu) (**email:** udit.raha@iitg.ac.in)

Indian Institute of Technology Guwahati

Jan 2020

Syllabus

- Pre Mid-Term Lectures (Myself & Bibhas Majhi)
  1. Vector Calculus: Gradient, Divergence and Curl; Line, Surface and Volume integrals; Gauss's divergence theorem and Stokes' curl theorem in Cartesian, Spherical polar, and Cylindrical polar coordinates; Dirac Delta function.
  2. Electrostatics: Gauss's law and its applications; Divergence and Curl of Electrostatic fields, Electrostatic Potential; Boundary Conditions; Work and Energy; Conductors and Capacitors; Laplace's equation: Solution by Method of Images & Variable Separable Method of solving PDE for Boundary Valued Problems involving Cartesian Coordinate Systems ONLY; Dielectric Media: Polarization, Bound Charges, Electric Displacement; Boundary conditions in dielectrics; Energy and Forces in dielectrics.

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- Post Mid-Term Lectures (Sovan Chakraborty & Debasish Bora)
  1. **Magnetostatics:** Lorentz force. BiotSavart and Ampere's laws and their applications. Divergence and Curl of Magnetostatic fields, Magnetic Vector Potential. Force and torque on a magnetic dipole. Magnetic materials. Magnetization, Bound currents. Boundary conditions.
  2. **Electrodynamics:** Ohm's law. Motional EMF, Faraday's law. Lenz's law. Self and Mutual inductance. Energy stored in magnetic field. Maxwell's equations. Continuity Equation, Poynting Theorem, Wave solution of Maxwell's Equations.
  3. **Electromagnetic Waves:** Polarization, reflection and transmission at oblique incidences

Reading Material

- Textbook:
  1. D. J. Griffiths, Introduction to Electrodynamics, 4^{th} Ed. Prentice-Hall (1995).
- References:
  1. N. Ida, Engineering Electrodynamics, Springer (2005).
  2. M. N. O. Sadiku, Elements of Electromagnetics, Oxford (2006)
  3. Feynman, Leighton, and Sands, The Feynman Lectures on Physics, Vol. II, Norosa Publishing House (1998).
  4. I. S. Grant and W. R. Phillips, Electromagnetism, John Wiley, (1990).

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- Web page for Lecture Slides & Tutorials.
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- For further queries you may consult me at my **Office:** Physics Dept. 3^{rd} Floor Room No. 6.
- For appointment first email me at udit.raha@iitg.ac.in

PH 102 Lecture Classes (in L2) & Tutorials Time-Table

- Tuesday Tutorial:
  7:55 AM - 8:55 AM (in respective Tutorial Groups)
- Wednesday Lecture:
  11:00 AM - 11:55 AM (Div III) & 4:00 PM - 4:55 PM (Div I)
- Thursday Lecture:
  11:00 AM - 11:55 AM (Div III) & 4:00 PM - 4:55 PM (Div I)

Assessments

- All examinations will be primarily subjective type with both long and short answer type question

Examinations	Dates	Marks
Quiz-I	February, 4	10
Mid-Semester	March, 2	30
Quiz-II	To be announced	10
End-Semester	May, 6	50

REQUEST: Please regularly attend **ALL** Lectures and Tutorial classes

Note: 75% attendence is the minimum passing criterion for this course

Preliminary Vector (Analysis) Calculus

1. Scalar and Vectors Fields: Definitions and Examples.
2. Differential Calculus of Fields: Ordinary, Partial and Total Derivatives.
3. Differential Operators: Gradient, Divergence, Curl (Rotation/Rot) and Laplacian.
4. Integrals in Vector Analysis: Line Integrals, Surface (Flux) Integrals, and Volume Integrals.
5. Fundamental Theorems: Gradient Theorem, Gauss's Divergence Theorem and Stokes' Curl Theorem
6. Orthogonal Coordinate Systems: Cartesian, Spherical polar, and Cylindrical polar coordinates.
7. The Dirac-Delta function: Definitions and Applications.

SO LET'S GET STARTED ...

Concept of Fields

While describing extended objects in physics that fill up some space or regions of space, we need to define abstract "objects" called FIELDS. With each point of the space, if we associate *scalar properties* we need SCALAR FIELDS, or if we associate *vector properties* we need VECTOR FIELDS.

Definition

If we consider a function f defined over a multi-D domain, i.e., with $m \geq 1$

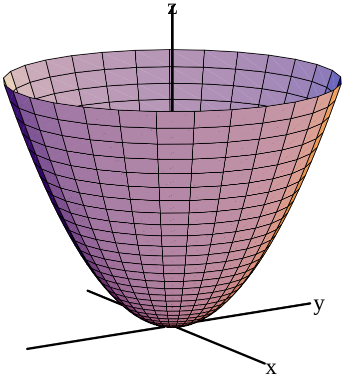
$$f : \mathbb{R}^m \to \mathbb{R}^n,$$

then for $n = 1$, the set of functional values of f at all points P in the space, i.e., $S = \{f(P)|\ f : \mathbb{R}^m \to \mathbb{R},\ \forall P \in \mathbb{R}^m\}$ defines a *scalar field* over $\mathbb{R}^m$, otherwise for $n > 1$, S defines a *vector field* over $\mathbb{R}^m$.

Examples

- Temperature T, Pressure P and Density ρ functions of a fluid (scalar fields)
- Potential functions ϕ, e.g., Gravitational, Electrostatic, etc. (scalar fields)
- Position vector $\mathbf{r}$ of a particle (vector field)
- Velocity vector $\mathbf{v}$ of a rotating body, or of a streamline fluid flow (vector field)
- Forces fields $\mathbf{F}$, e.g., Gravitational, Electrostatic, etc. (vector fields)

Scalar Field (2D)



Example

A scalar field over $\mathbb{R}^2$ given by the set S and defined by the function $f(x, y)$:

$$S = \left\{ f(x, y) \middle| \, f = x^2 + y^2, \, \forall (x, y) \in \mathbb{R}^2 \right\}$$

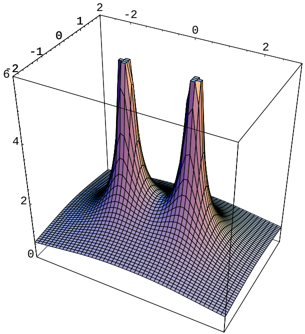
2D Scalar Field in Physics

Example

The **Electrostatic Potential** function $V(x,y)$ for two *identical* point charges Q at $(1,0,0)$ and $(-1,0,0)$ in xy-plane $(z=0)$ defines a scalar field in $\mathbb{R}^2$

$$V(x,y) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-1)^2+y^2}} + \frac{1}{\sqrt{(x+1)^2+y^2}}\right)$$

Countour plot displays the family of EQUIPOTENTIALS



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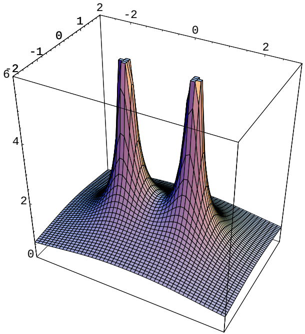
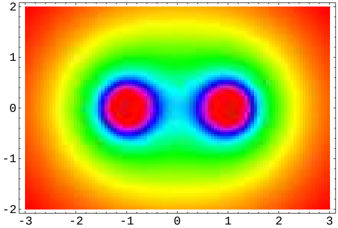
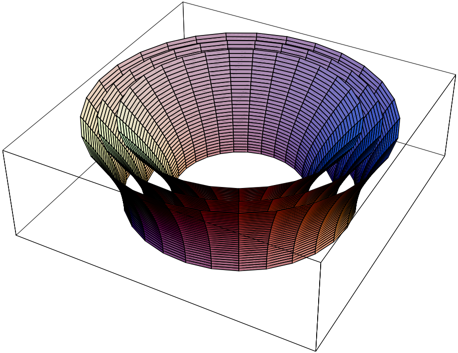




Scalar Field (3D)

Example



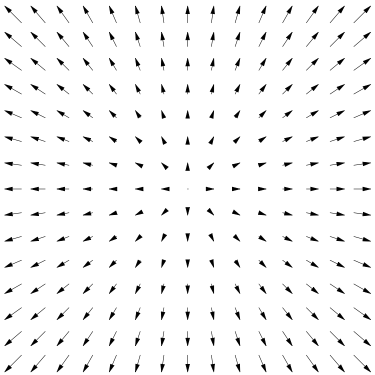
$L \rightarrow$ Family of Coaxial level surfaces in $\mathbb{R}^3$ defined by $f(X, Y, Z) = const.$

$$\begin{aligned} S &= \{f(x,y,z) | \, f = x^2 + y^2 - z^2, \, (x,y,z) \in \mathbb{R}^3\} \\ L &= \{f(X,Y,Z) = X^2 + Y^2 - Z^2 = const. | \, (X,Y,Z) \subset \mathbb{R}^3\} \end{aligned}$$

Vector Fields (2D)

Example

Vector fields defined in terms of vector-valued functions $\mathbf{V} \in \mathbb{R}^2$

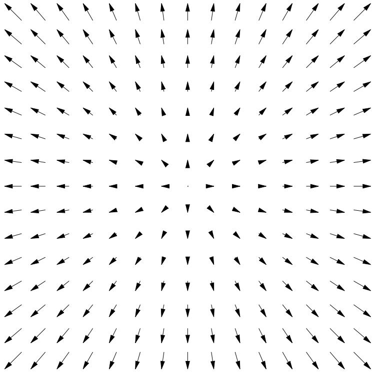


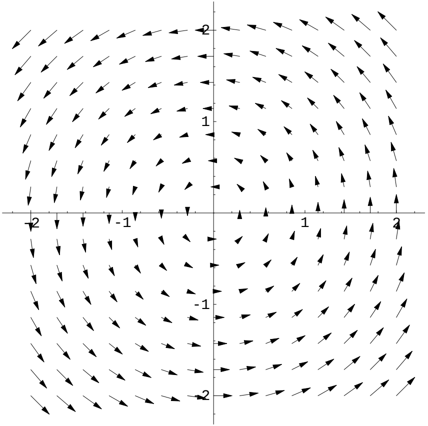
$$S_1 = \left\{ \mathbf{V}_1(x, y) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}},\ (x, y) \in \mathbb{R}^2 \right\}$$

Vector Fields (2D)

Example

Vector fields defined in terms of <u>vector-valued</u> functions $\mathbf{V} \in \mathbb{R}^2$





$$S_1 = \left\{ \mathbf{V}_1(x,y) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}},\ (x,y) \in \mathbb{R}^2 \right\}$$

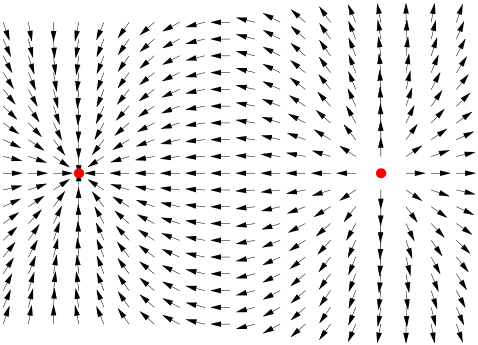
$$S_2 = \left\{ \mathbf{V}_2(x,y) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}},\ (x,y) \in \mathbb{R}^2 \right\}$$

2D Vector Field in Physics

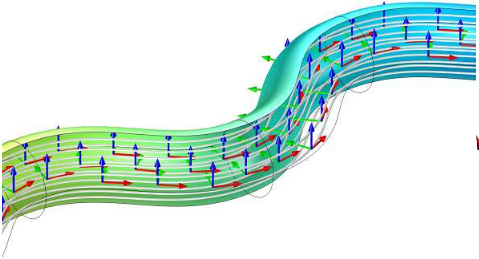
Example

The vector function $\mathbf{E}(x, y)$ represents the **Electric Field** due to *opposite* point charges Q and $-Q$ at $(1, 0, 0)$ and $(-1, 0, 0)$, respectively, in xy-plane $(z = 0)$

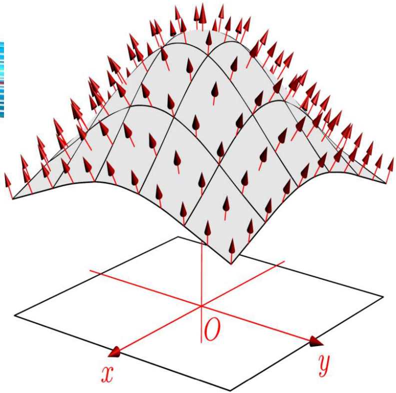
$$\mathbf{E}(x, y)|_{z=0} = \frac{Q}{4\pi\epsilon_0}\left(\frac{(x-1)\,\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{\left((x-1)^2 + y^2\right)^{3/2}} - \frac{(x+1)\,\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{\left((x+1)^2 + y^2\right)^{3/2}}\right)$$



Physical Vector Fields (3D)



Pressure gradient field and **Velocity field** for streamline liquid flow



Flux across a surface/membrane

Example

Vector fields in $\mathbb{R}^3$ are defined in terms of <u>vector-valued</u> functions in $\mathbb{R}^3$

Single Variable Calculus: Ordinary Derivatives

Definition

Suppose $\mathbf{V} : \mathbb{R} \to \mathbb{R}^3$ be a *vector valued function* of a single variable $t \in \mathbb{R}$. Then, the *derivative* of $\mathbf{V}$ with respect to (w.r.t.) t is given by

$$\mathbf{V}'(t) = \frac{d\mathbf{V}(t)}{dt} = \lim_{\delta t \to 0} \frac{\mathbf{V}(t + \delta t) - \mathbf{V}(t)}{\delta t}$$

if the above limit exits. Equivalently, with components: $V_1, V_2, V_3 \in \mathbb{R}$, each being independent function of t:

$$V_i'(t) = \frac{dV_i(t)}{dt} = \lim_{\delta t \to 0} \frac{V_i(t + \delta t) - V_i(t)}{\delta t}, \quad i = 1, 2, 3.$$

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Since $\mathbf{V}'(t)$ is itself a vector, we can consider the *second derivative* with respect to t, if it exists, i.e., $\frac{d\mathbf{V}'(t)}{dt}$, and denoted by the symbol $\frac{d^2\mathbf{V}(t)}{dt^2}$.

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Definition

In particular we can define a DIFFERENTIAL of the vector function $\mathbf{V}$:

$$d\mathbf{V}(t) \equiv \left(\frac{d\mathbf{V}(t)}{dt}\right) dt = \mathbf{V}'(t)\, dt.$$

This is called the CHAIN RULE of ordinary derivatives.

Multivariate Calculus: Partial Derivatives

Definition

Suppose $\mathbf{V}$ be a continuous *vector function* of several independent variables, say, $u, v, t \in \mathbb{R}$. Then, the PARTIAL derivative of $\mathbf{V}$ w.r.t., say t, is given by

$$\mathbf{V}_t(u, v, t) \equiv \frac{\partial \mathbf{V}(u, v, t)}{\partial t} = \lim_{\delta t \to 0} \frac{\mathbf{V}(u, v,\ t + \delta t) - \mathbf{V}(u, v, t)}{\delta t}$$

if the limit exists.

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if the limit exists.

Likewise, the partial derivative w.r.t. v:

$$\mathbf{V}_v(u, v, t) \equiv \frac{\partial \mathbf{V}(u, v, t)}{\partial v} = \lim_{\delta v \to 0} \frac{\mathbf{V}(u, \ v + \partial v, \ t) - \mathbf{V}(u, v, t)}{\delta v}$$

Thus, the *partial derivatives* signify how rapidly the (vector) function varies when one of the variables in the argument is changed by a *infinitesimal* amount, when the **other variables held fixed**. Different notations may be used for the partial derivative w.r.t. t, e,g., $\mathbf{V}_t$, $\mathbf{V}'_t$, $\partial_t \mathbf{V}$, $\frac{\partial \mathbf{V}}{\partial t}$, $\frac{\partial}{\partial t}\mathbf{V}$,...

Multivariate Calculus: Partial Derivatives

Definition

Suppose $\mathbf{V}$ be a continuous *vector function* of several independent variables, say, $u, v, t \in \mathbb{R}$. Then, the PARTIAL derivative of $\mathbf{V}$ w.r.t., say t, is given by

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Higher (mixed) partial derivatives:

$$\mathbf{V}_{tt} = \frac{\partial^2 \mathbf{V}}{\partial t^2}, \quad \mathbf{V}_{vu} = \frac{\partial^2 \mathbf{V}}{\partial u\, \partial v} = \frac{\partial^2 \mathbf{V}}{\partial v\, \partial u} = \mathbf{V}_{uv}, \quad \mathbf{V}_{uvt} = \frac{\partial^3 \mathbf{V}}{\partial t\, \partial v\, \partial u} = \mathbf{V}_{vtu} = \mathbf{V}_{utv} = \cdots$$

CHAIN RULE: Total Differential & Derivative

Definition

Suppose $\mathbf{V}(x, y, z)$ be a smooth *vector valued function* of independent variables $x, y, z \in \mathbb{R}$ with continuous partial derivatives. Then, the TOTAL DIFFERENTIAL of $\mathbf{V}$ is given by the CHAIN RULE:

$$d\mathbf{V}(x, y, z) = \left(\frac{\partial \mathbf{V}}{\partial x}\right)_{y,z\to const.} dx + \left(\frac{\partial \mathbf{V}}{\partial y}\right)_{x,z\to const.} dy + \left(\frac{\partial \mathbf{V}}{\partial z}\right)_{x,y\to const.} dz$$

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Definition

TOTAL DERIVATIVE of $\mathbf{V}$ which may implicitely or explicitely depends on another variable, say, $t \in \mathbb{R}$, assuming the variables, x, y, z to depend on t:

$$\begin{aligned}\frac{d\mathbf{V}\left(x(t), y(t), z(t)\right)}{dt} &= \left(\frac{\partial \mathbf{V}}{\partial x}\right)_{y,z} \frac{dx(t)}{dt} + \left(\frac{\partial \mathbf{V}}{\partial y}\right)_{x,z} \frac{dy(t)}{dt} + \left(\frac{\partial \mathbf{V}}{\partial z}\right)_{x,y} \frac{dz(t)}{dt} \\ &\equiv \mathbf{V}_x \frac{dx(t)}{dt} + \mathbf{V}_y \frac{dy(t)}{dt} + \mathbf{V}_z \frac{dz(t)}{dt} : \textbf{ Implicitely}\end{aligned}$$

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$$\equiv \mathbf{V}_x \frac{dx(t)}{dt} + \mathbf{V}_y \frac{dy(t)}{dt} + \mathbf{V}_z \frac{dz(t)}{dt}: \textbf{ Implicitely}$$

$$\textbf{Explicitely}: \frac{d\mathbf{V}\left(x(t), y(t), z(t), t\right)}{dt} = \mathbf{V}_x \frac{dx(t)}{dt} + \mathbf{V}_y \frac{dy(t)}{dt} + \mathbf{V}_z \frac{dz(t)}{dt} + \mathbf{V}_t$$

Total Spatial Derivatives in $\mathbb{R}^2$ & $\mathbb{R}^3$ $\rightarrow$ Directional Derivatives

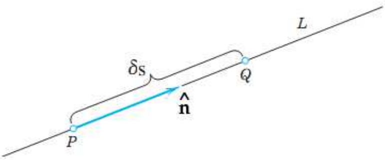
Definition

Consider the continuous scalar function $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$, defining a scalar field in $\mathbb{R}^3$. Let $\mathbf{r} \in \mathbb{R}^3$ be a point P and $\hat{\mathbf{n}} \in \mathbb{R}^3$ be an unit vector in a given direction. Consider a *ray* L from P in the direction $\hat{\mathbf{n}}$ and a second point Q on this ray at a small distance δs from P. Then,

$$D_{\hat{\mathbf{n}}}\phi(\mathbf{r}) \equiv \frac{d\phi(\mathbf{r})}{ds} = \lim_{\delta s \to 0} \frac{\phi(\mathrm{Q}) - \phi(\mathrm{P})}{\delta s} = \lim_{\delta s \to 0} \frac{\phi(\mathbf{r} + \delta s\, \hat{\mathbf{n}}) - \phi(\mathbf{r})}{\delta s}$$

is called the DIRECTIONAL derivative of ϕ at $P(\mathbf{r})$ in the direction of $\hat{\mathbf{n}}$.

Thus, $\frac{d\phi}{ds}$ yields the rate of change of ϕ at point P in the direction of $\hat{\mathbf{n}}$



Directional Derivative

Cartesian system $\rightarrow P(x, y, z)$ and $Q(x+\delta x, y+\delta y, z+\delta z)$

$$\vec{PQ} \equiv \delta \mathbf{r} = \delta s\, \hat{\mathbf{n}} = \delta x(s)\, \hat{\mathbf{i}} + \delta y(s)\, \hat{\mathbf{j}} + \delta z(s)\, \hat{\mathbf{k}}$$

RECALL $\Rightarrow$ Total differential formula with $\delta\phi \rightarrow d\phi$ for $\delta s \rightarrow 0$:

$$\begin{aligned}\delta\phi = \phi(\mathrm{Q}) - \phi(\mathrm{P}) &= \phi(x+\delta x(s),\ y+\delta y(s),\ z+\delta z(s)) - \phi(x,y,z) \\ &= \frac{\partial\phi}{\partial x}\,\delta x(s) + \frac{\partial\phi}{\partial y}\,\delta y(s) + \frac{\partial\phi}{\partial z}\,\delta z(s)\end{aligned}$$

Then the directional derivative of $\phi(\mathbf{r})$ is

$$\frac{d\phi(x,y,z)}{ds} = \lim_{\delta s \to 0} \frac{\phi(\mathrm{Q}) - \phi(\mathrm{P})}{\delta s} = \frac{\partial\phi}{\partial x}\left(\frac{dx(s)}{ds}\right) + \frac{\partial\phi}{\partial y}\left(\frac{dy(s)}{ds}\right) + \frac{\partial\phi}{\partial z}\left(\frac{dz(s)}{ds}\right)$$

$$= \left(\hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z}\right) \cdot \left(\frac{d\mathbf{r}(s)}{ds}\right)$$

Directional Derivative

Cartesian system $\rightarrow$ $P(x, y, z)$ and $Q(x + \delta x, y + \delta y, z + \delta z)$

$$\vec{PQ} \equiv \delta \mathbf{r} = \delta s\, \hat{\mathbf{n}} = \delta x(s)\, \hat{\mathbf{i}} + \delta y(s)\, \hat{\mathbf{j}} + \delta z(s)\, \hat{\mathbf{k}}$$

RECALL $\Rightarrow$ Total differential formula with $\delta\phi \rightarrow d\phi$ for $\delta s \rightarrow 0$:

$$\delta\phi = \phi(\mathrm{Q}) - \phi(\mathrm{P}) = \phi(x + \delta x(s),\, y + \delta y(s),\, z + \delta z(s)) - \phi(x, y, z)$$
$$= \frac{\partial \phi}{\partial x}\, \delta x(s) + \frac{\partial \phi}{\partial y}\, \delta y(s) + \frac{\partial \phi}{\partial z}\, \delta z(s)$$

Then the directional derivative of $\phi(\mathbf{r})$ is

$$\frac{d\phi(x, y, z)}{ds} = \lim_{\delta s \to 0} \frac{\phi(\mathrm{Q}) - \phi(\mathrm{P})}{\delta s} = \frac{\partial \phi}{\partial x} \left(\frac{dx(s)}{ds} \right) + \frac{\partial \phi}{\partial y} \left(\frac{dy(s)}{ds} \right) + \frac{\partial \phi}{\partial z} \left(\frac{dz(s)}{ds} \right)$$

$$= \left(\hat{\mathbf{i}} \frac{\partial \phi}{\partial x} + \hat{\mathbf{j}} \frac{\partial \phi}{\partial y} + \hat{\mathbf{k}} \frac{\partial \phi}{\partial z} \right) \cdot \left(\frac{d\mathbf{r}(s)}{ds} \right) \equiv \mathrm{grad}\, \phi \cdot \hat{\mathbf{n}}$$

The vector function grad ϕ is called GRADIENT and defines a vector field.

Gradient Operator ∇ ("nabla")

$$\nabla \equiv \mathrm{grad} = \left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right)$$

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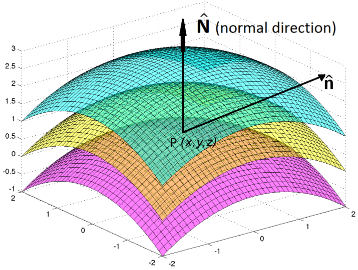
Consider a differentiable scalar function $\phi : \mathbb{R}^3 \to \mathbb{R}$, with $\phi(x, y, z) = k \Rightarrow \text{constant},\ k \in \mathbb{R}$. This represents a family of non-intersecting LEVEL SURFACES in 3D space for different values of the constant k.

Corollary

What is the direction of gradient of ϕ at a given point P, i.e., $\nabla\phi(P)$?



Geometric Interpretation of ∇

Consider the level surface given by $\phi(x, y, z) = k$ containing point P(x, y, z).
For any direction, say, $\hat{\mathbf{n}}$ from P, consider the **total differential** $d\phi$:

$$d\phi(P) = \left(\frac{\partial \phi}{\partial x}\right)_P dx + \left(\frac{\partial \phi}{\partial y}\right)_P dy + \left(\frac{\partial \phi}{\partial z}\right)_P dz \qquad : \text{chain rule}$$

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&= |\nabla\phi(P)|\,|\hat{\mathbf{n}}|\, ds \cos\theta \\
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where θ is the angle between $\nabla\phi(P)$ and the direction $\hat{\mathbf{n}}$.

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The gradient of a scalar function ϕ at any given point P yields the maximum value of its directional derivative at that point, and its direction obviously points along the normal direction $\hat{\mathbf{N}}$ for the level surface $\phi(x,y,z)=k$, i.e., the direction with the steepest rate of change of ϕ at that point.

Product Identities (Gradient)

If ϕ and ψ are differentiable scalar fields, and if $\mathbf{A}$ and $\mathbf{B}$ are differentiable vector fields, then $\phi\,\psi$ and $\mathbf{A}\cdot\mathbf{B}$ are both scalar fields. Also if k is a constant scalar and n is any integer, then the following **product identities** hold:

- $\nabla(k\,\phi) = k\,\nabla\phi$
- $\nabla\left(\phi\,\psi\right) = \phi\nabla\psi + \psi\nabla\phi$
- $\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}$
- $\nabla\phi^n = n\phi^{n-1}\nabla\phi$
- $\nabla\left(\mathbf{A}\cdot\mathbf{B}\right) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$

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$$\begin{aligned}
(\mathbf{A}\cdot\nabla)\mathbf{B} &= \left[(A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}})\cdot\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\right]\mathbf{B} \\
&= \left(A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}\right)\mathbf{B} \\
&= A_1\frac{\partial \mathbf{B}}{\partial x} + A_2\frac{\partial \mathbf{B}}{\partial y} + A_3\frac{\partial \mathbf{B}}{\partial z} \\
&= A_1\frac{\partial}{\partial x}(B_1\hat{\mathbf{i}} + B_2\hat{\mathbf{j}} + B_3\hat{\mathbf{k}}) + A_2\cdots
\end{aligned}$$

Product Identities for Gradient (contd.)

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$

Product Identities for Gradient (contd.)

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Hint: Proof of the last Identity

$$\begin{aligned}\nabla(\mathbf{A}\cdot\mathbf{B}) &= \nabla(A_xB_x + A_yB_y + A_zB_z)\\ &= (A_x\nabla B_x + A_y\nabla B_y + A_z\nabla B_z) + (B_x\nabla A_x + B_y\nabla A_y + B_z\nabla A_z)\end{aligned}$$

The x-component of the first bracket:

$$\begin{array}{cccc} & +A_x\partial_x B_x & +A_y\partial_x B_y & +A_z\partial_x B_z \\ & +A_y\partial_y B_x & -A_y\partial_y B_x & \\ & +A_z\partial_z B_x & & -A_z\partial_z B_x \\ \hline (\mathbf{A}\cdot\nabla)B_x & & +A_y(\nabla\times\mathbf{B})_z & -A_z(\nabla\times\mathbf{B})_y \end{array}$$

$$= (\mathbf{A}\cdot\nabla)B_x + (\mathbf{A}\times(\nabla\times\mathbf{B}))_x$$