Physics II (PH 102)

Electromagnetism (Lecture 7)

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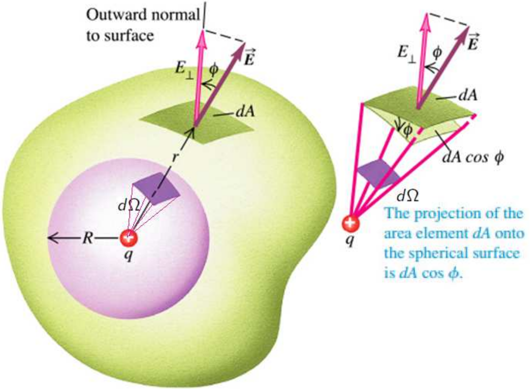
Concept of Electric Flux

Definition

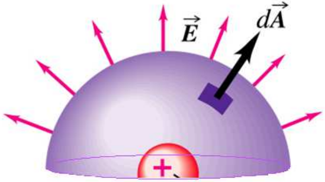
Let S be any arbitrary simple surface (open or closed) and $\mathbf{E}$ is the Electric field in the region containing S. Then, the *total flux of* $\mathbf{E}$ *through the surface* is defined as the surface integral of the outward normal component of $\mathbf{E}$ on S:

$$\phi_S = \iint_S \mathbf{E}\cdot d\mathbf{A} = \iint_S E\, dA \cos\phi = \frac{q}{4\pi\epsilon_0} \iint_S \left(\frac{dA\cos\phi}{r^2}\right) = \frac{q}{4\pi\epsilon_0} \underbrace{\int d\Omega}_{\text{Solid angle}} = \frac{q}{\epsilon_0},$$

Thus, the result is independent of the specific geometry of the surface S.



Electric Flux due to Point Charge



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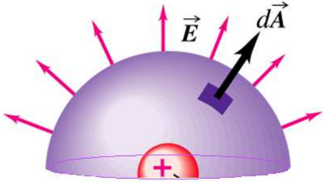

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Let $d\mathbf{A} = R^2 \sin\theta d\theta d\phi\, \hat{\mathbf{r}}$ be an elementary area on the hemesphere at $\mathbf{r} = R\,\hat{\mathbf{r}}$, where $|\mathbf{r}| = R$, and unit normal to dA is $\hat{\mathbf{r}}$. Hence, the Electric flux is

$$\phi_S = \iint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{A} \;\; ; \quad \mathbf{E}(\mathbf{r} = R\,\hat{\mathbf{r}}) = \frac{q}{4\pi\epsilon_0}\left(\frac{\hat{\mathbf{r}}}{R^2}\right)$$

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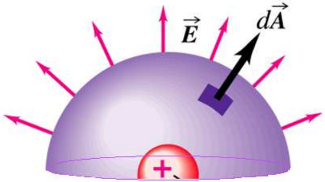

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&= \frac{q}{4\pi\epsilon_0}\int_0^{\pi/2}\int_0^{2\pi}\left(\frac{\hat{\mathbf{r}}}{R^2}\right)\cdot\hat{\mathbf{r}}\, R^2 \sin\theta d\theta d\phi \\
&= \frac{q}{2\epsilon_0}
\end{aligned}
$$

Gauss's Integral Law ← Differential Law

Let $\mathbf{E}$ be the Electric field defined over a volume V with volume charge density ρ, then using the **differential form of Gauss's law** we have

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where Q_{enclosed} is the total charge enclosed within V.

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$$\iiint_V \nabla \cdot \mathbf{E}(\mathbf{r})\, dv = \frac{Q_{\text{enclosed}}}{\epsilon_0} = \oiint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{A} = \phi_S$$

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Gauss's Integral Law:

The surface integral of the outward normal component of the Electric field, i.e., the total normal Electric flux, ϕ_S over a <u>closed surface</u> S enclosing a total charge Q_{enclosed} is equal the ratio $Q_{\mathrm{enclosed}}/\epsilon_0$, and this result is independent of the specific geometry (shape) of the surface. Mathematically,

$$\phi_S = \oiint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

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- First contruct a fully closed GAUSSIAN SURFACE. Open surfaces, like discs, can not enclose charge in a 3D volume.
- The surface must include the point where the Electric field is calculated.
- The surface is chosen in such a way that for every point on that surface the Electric field **E** is constant.

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- 4 types of symmetries can be exploited: (1) Spherical, (2) Cylindrical (3) Cubical and (4) Planar. Accordingly, the Gaussian surfaces are constructed **spherical**, **cylindrical**, **cubical** and **pillbox** shaped.

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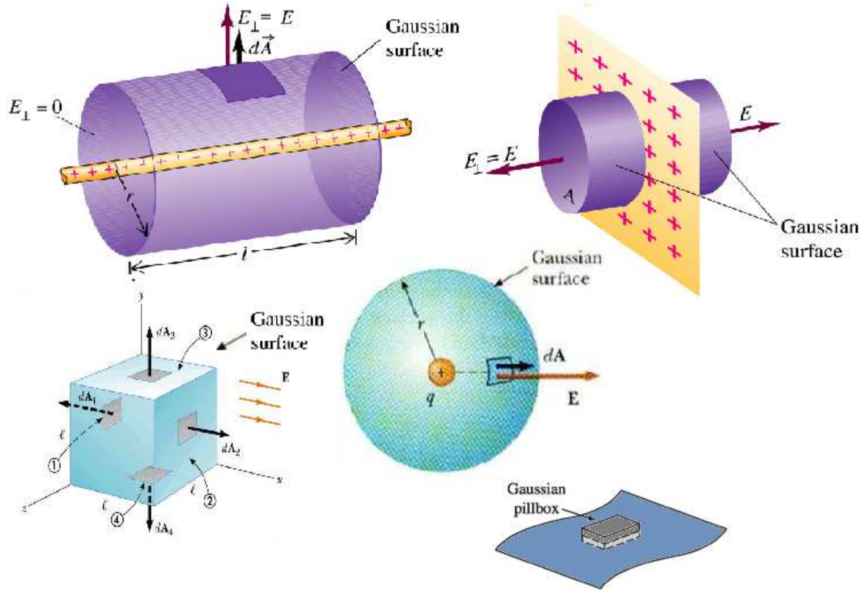
- 4 types of symmetries can be exploited: (1) Spherical, (2) Cylindrical (3) Cubical and (4) Planar. Accordingly, the Gaussian surfaces are constructed **spherical**, **cylindrical**, **cubical** and **pillbox** shaped.
- Surface integration drastically simplifies, since **E** being constant can be taken outside the integral.

$$\phi_s = \oiint_S \mathbf{E} \cdot d\mathbf{A} = \mathbf{E} \cdot \hat{n} \oiint_S dA = E_n(\text{Surface Area})$$

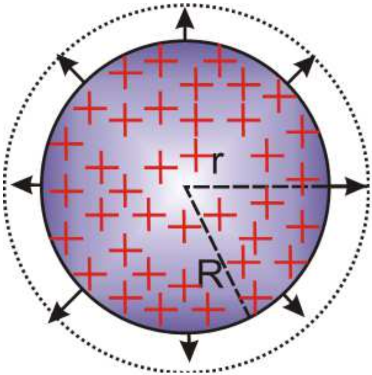
Typical Gaussian Surfaces

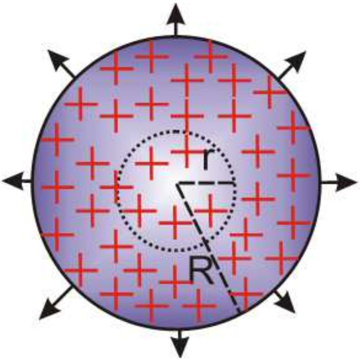


Applications of Gauss's Law

Example

Consider a uniformly charged *insulating* sphere of radius R and charge Q. Calculate Electric fields both outside and inside the sphere.





Outside the charged Sphere ($r > R$)

Construct a Gaussian surface S_{out} of raduis r outside the charged sphere with the same center, then the total enclosed charge is Q. Thus,

$$\begin{aligned}
\oiint_{S_{\text{out}}} \mathbf{E} \cdot d\mathbf{A} &= |\mathbf{E}| \oiint_{S_{\text{out}}} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dA = |\mathbf{E}| \, 4\pi r^2 = \frac{Q}{\epsilon_0} \\
\mathbf{E}(r) &= \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}.
\end{aligned}$$

Inside the charged Sphere ($r < R$)

Construct a Gaussian surface S_{in} of raduis r inside the charged sphere with the same center. The amount of charge enclosed is

$$q_{\text{enclosed}} = \frac{Q}{\frac{4}{3}\pi R^3} \left(\frac{4}{3} \pi r^3 \right) = \frac{Q \, r^3}{R^3}.$$

Thus,

$$\begin{aligned}
\oiint_{S_{\text{in}}} \mathbf{E} \cdot d\mathbf{A} &= |\mathbf{E}| \oiint_{S_{\text{in}}} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dA = |\mathbf{E}| \, 4\pi r^2 = \frac{1}{\epsilon_0} \left(\frac{Q \, r^3}{R^3} \right) \\
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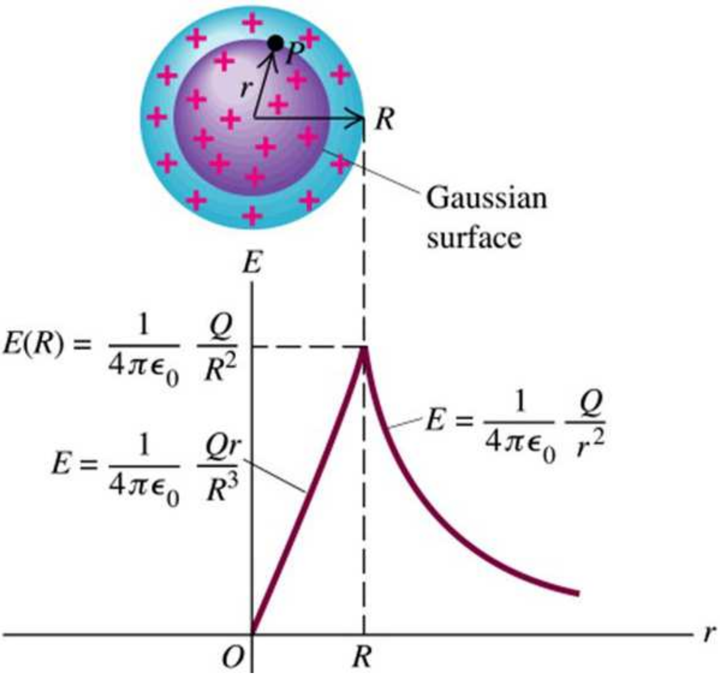
\end{aligned}$$


P
r
R
Gaussian surface
E
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O
R
r

Curl of Electric Field due to a Point Charge

Suppose a Point Source Charge of magnitude q is placed at the origin. Electric field at a Field point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}.$$

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Now curl of Electric field in Cartesian system will be

$$\nabla \times \mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ (x/r^3) & (y/r^3) & (z/r^3) \end{vmatrix}$$

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[\nabla \times \mathbf{E}(\mathbf{r})]_x &= \frac{q}{4\pi\epsilon_0}\left[\partial_y\left(\frac{z}{r^3}\right) - \partial_z\left(\frac{y}{r^3}\right)\right] \\
&= \frac{q}{4\pi\epsilon_0}\left[\left(-\frac{3yz}{r^5}\right) - \left(\frac{-3zy}{r^5}\right)\right] = 0.
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Similarly, the other components: $[\nabla \times \mathbf{E}(\mathbf{r})]_y = [\nabla \times \mathbf{E}(\mathbf{r})]_z = 0$. Thus,

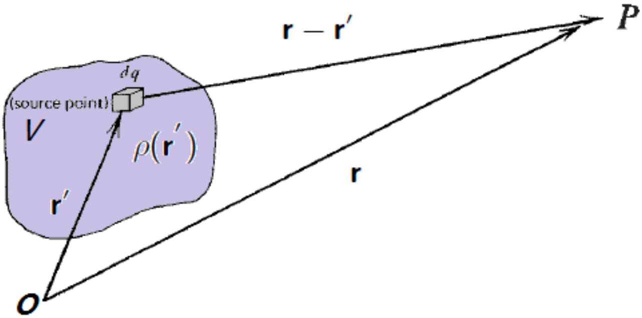
$$\nabla \times \mathbf{E}(\mathbf{r}) = 0$$

This is a general result in ELECTROSTATICS, but not necesarily in ELECTRODYNAMICS where $\nabla \times \vec{\mathcal{E}}(\mathbf{r}, t) \neq 0$

Curl of Electric Field due to a Continuous Volume Distribution

Now we extend the result to arbitrary Volume Charge Distribution V with volume density ρ. Electric field at the target point P is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\rho(\mathbf{r}^{'}) \left(\mathbf{r} - \mathbf{r}^{'}\right)}{|\mathbf{r} - \mathbf{r}^{'}|^3} dv^{'}$$



Curl of Electric Field due to a continuous distribution (contd.)

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Here we are interested in the curl with respect to TARGET POINT variable $\mathbf{r}$:

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$$\left(\nabla \times \frac{\left(\mathbf{r}-\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|^3}\right)_x = \left(-3(z-z')\frac{(y-y')}{|\mathbf{r}-\mathbf{r}'|^5} + 3(y-y')\frac{(z-z')}{|\mathbf{r}-\mathbf{r}'|^5}\right) = 0$$

The other components similarly vanish.

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Curl of Electric field $\mathbf{E}$ is ALWAYS zero in electrostatics, and therefore $\mathbf{E}$ can be derived from the gradient of an arbitrary scalar field $\phi(\mathbf{r})$:

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0 \quad \Longrightarrow \quad \mathbf{E}(\mathbf{r}) = \nabla\phi(\mathbf{r})$$

Note: In electrodynamics, for *time varying* e.m. fields, $\nabla \times \vec{\mathcal{E}}(\mathbf{r},t) \neq 0$.

Concept of Electrostatic Potential Function

Electric field **E** is by convention taken as the negative gradient of the Electrostatic Potential V .

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Integrating from a reference point **ref** to the point **r** along arb. path C:

$$\int_C \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}' = -\int_{\mathbf{ref}}^{\mathbf{r}} \nabla V(\mathbf{r}') \cdot d\mathbf{r}' = -\int_{\mathbf{ref}}^{\mathbf{r}} dV = -V(\mathbf{r}) + \cancelto{0}{V(\mathbf{ref})}$$

The **ref** point is so chosen that the Potential at that location is zero, which is conventionally taken at $\mathbf{ref} = \infty$ for finite charge distributions and for infinitely extended charge distributions the **ref** point may be chosen arbitrarily.

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Definition

The Electrostatic Potential at any point in an existing electric field is equal to the work done by an external agent against the repulsive electric forces in carrying a unit positive test charge from infinity to that point, i.e.,

$$
V(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \mathbf{E}(\mathbf{r}')\cdot d\mathbf{r}'
$$

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- Sign Convension: A positive Potential implies *work done by the external agent on the Electrostatic field*, and a negative Potential implies *work done by the Electrostatic field*.
- Corollary from **Stokes' Theorem**: Since $\nabla \times \mathbf{E}(\mathbf{r}) = 0$, the circulation of the Electric field about any closed path is zero, and so is the net work done $\Longrightarrow$ conservative nature of Electric field.

$$\iint_S [\nabla \times \mathbf{E}(\mathbf{r})] \cdot d\mathbf{S} = \oint_L \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r} = 0.$$

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- Corollary from **Fundamental Gradient Theorem**: The line integral of the Electric field is path-independent and depends only on the end points $\implies$ *potential difference* between the given end points is uniquely given by

$$\int_{\mathbf{r}=\mathbf{a}}^{\mathbf{r}=\mathbf{b}} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r} = -\int_{\mathbf{a}}^{\mathbf{b}} \nabla V(\mathbf{r}) \cdot d\mathbf{r} = -\int_{\mathbf{a}}^{\mathbf{b}} dV = V(\mathbf{a}) - V(\mathbf{b}) \equiv \Delta V_{ab}.$$

Electrostatic Potential: Facts (contd.)

- **Linear Superposition Principle** holds for Potentials: The total Electrostatic Potential at any point is the sum of the Electrostatic Potentials due to all the source charges/charge distributions separately, i.e.,

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- **Linear Superposition Principle** holds for Potentials: The total Electrostatic Potential at any point is the sum of the Electrostatic Potentials due to all the source charges/charge distributions separately, i.e.,

$$V_{\mathrm{Total}} = V_1 + V_2 + \cdots = \sum_i V_i$$

- General Charge Distribution: For a localized charge distribution with volume, surface and linear densities, ρ, σ, λ, respectively, as well as discrete point charges q_i, the resulting Electrostatic Potential is the **Superposition of Potentials** due to the independent distributions:

$$\begin{aligned} V(\mathbf{r}) \quad &= \quad \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{|\mathbf{r}-\mathbf{r}_i^{'}|} + \frac{1}{4\pi\epsilon_0}\int_C \frac{\lambda(\mathbf{r}^{'})}{|\mathbf{r}-\mathbf{r}^{'}|}dl^{'} \\ &+ \frac{1}{4\pi\epsilon_0}\iint_S \frac{\sigma(\mathbf{r}^{'})}{|\mathbf{r}-\mathbf{r}^{'}|}dS^{'} + \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\mathbf{r}^{'})}{|\mathbf{r}-\mathbf{r}^{'}|}dv^{\prime}. \end{aligned}$$

Poisson's and Laplace's Equations

- We have seen that $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$ and $\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}$.
- Combining the two yields the **POISSON'S EQUATION**:

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{r}) &= \nabla \cdot [-\nabla V(\mathbf{r})] = -\nabla^2 V(\mathbf{r}) \\
\nabla^2 V(\mathbf{r}) &= -\frac{\rho(\mathbf{r})}{\epsilon_0}.
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- In regions where there are no charge distributions, $\rho = 0$, we obtain the **LAPLACE'S EQUATION**, $\nabla^2 V(\mathbf{r}) = 0$, e.g., in Cartesian System

$$\boxed{\nabla^2 V(\mathbf{r}) = \frac{\partial^2 V(x,y,z)}{\partial^2 x} + \frac{\partial^2 V(x,y,z)}{\partial^2 y} + \frac{\partial^2 V(x,y,z)}{\partial^2 z} = 0.}$$

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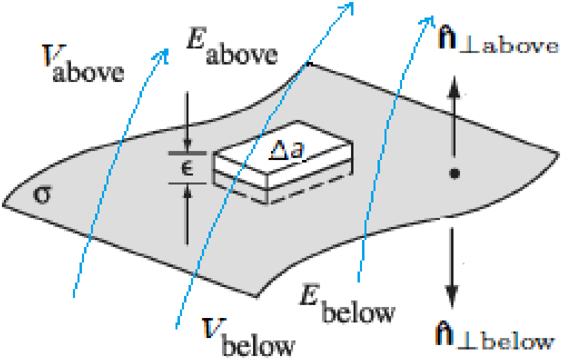
- **Boundary Valued Problems:**
  To obtain UNIQUE solutions to such 2^{nd} order *Partial Differential Equations* (PDEs), require specification of Potentials, Electric fields or charge configurations across *boundaries* or *interface* between different media, e.g., conductors or dielectrics with different physical properties. These specifications are termed as BOUNDARY CONDITIONS.

Boundary Conditions on **E** and V

Consider a arbitrarily shaped smooth *interface* with surface charge density σ. Construct a thin **wafer-like Gaussian pillbox** across the interface of vanishing thickness $\epsilon \to 0$ and infinitesimally small upper and lower "lid" surface areas Δa. The pillbox is contructed arbitrarily close, stradling to the interface so that the surface looks "locally flat" such that **E** is 'almost' constant on all its surfaces.



Here we shall find the relations between $\mathbf{E}_{above}$ & $\mathbf{E}_{below}$ and V_{above} & V_{below}.

Boundary Conditions on $\mathbf{E}$

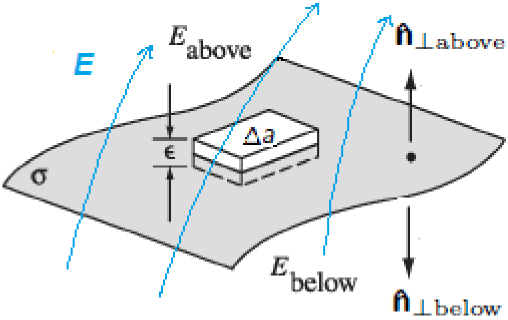
- Applying Gauss's law,

$$\lim_{\epsilon\to 0}\left[\ \oiint_{\text{pillbox}} \mathbf{E}\cdot d\mathbf{S}\right] = \left(\mathbf{E}_{\text{above}}\cdot\hat{\mathbf{n}}_{\perp\text{above}} + \mathbf{E}_{\text{below}}\cdot\hat{\mathbf{n}}_{\perp\text{below}}\right)\Delta a = \frac{1}{\epsilon_0}Q_{\text{encl}} = \frac{1}{\epsilon_0}\sigma\Delta a$$

$$\left(\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}\right)\cdot\hat{\mathbf{n}}_{\perp\text{above}} = \frac{\sigma}{\epsilon_0}$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}$$

Note: There is no contribution from the sides of the pillbox as $\epsilon \to 0$.

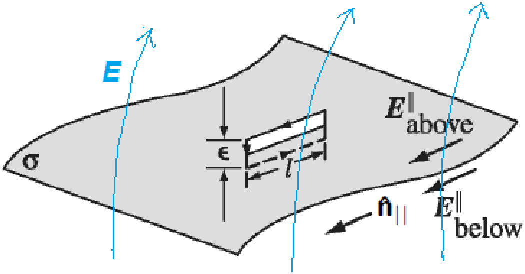


Boundary Conditions on E (contd.)

- Next consider a thin rectangular closed loop C stradling across the interface of vanishing ends $\epsilon \to 0$ and side lengths l. Then, since $\mathbf{E}$ is a conservative vector field

$$\begin{aligned}
\lim_{\epsilon \to 0} \left[\oint_C \mathbf{E} \cdot d\mathbf{l} \right] &= 0 \\
\left(\mathbf{E}_{\text{above}} \cdot \hat{\mathbf{n}}_{||} - \mathbf{E}_{\text{below}} \cdot \hat{\mathbf{n}}_{||} \right) l &= (\mathrm{E}^{||}_{\text{above}} - \mathrm{E}^{||}_{\text{below}}) l = 0 \\
\mathrm{E}^{||}_{\text{above}} &= \mathrm{E}^{||}_{\text{below}}
\end{aligned}$$

Note: The ends give vanishing contributions since $\epsilon \to 0$.



Boundary Conditions on **E** (contd.)

- The two previous results can be combined as:

$$
\begin{aligned}
E^{\perp}_{\mathrm{above}}(\mathbf{r}) - E^{\perp}_{\mathrm{below}}(\mathbf{r}) &= \frac{\sigma(\mathbf{r})}{\epsilon_0} \\
\mathrm{E}^{||}_{\mathrm{above}}(\mathbf{r}) &= \mathrm{E}^{||}_{\mathrm{below}}(\mathbf{r}) \\
\Longrightarrow \quad \mathbf{E}_{\mathrm{above}}(\mathbf{r}) - \mathbf{E}_{\mathrm{below}}(\mathbf{r}) &= \frac{\sigma(\mathbf{r})}{\epsilon_0}\hat{\mathbf{n}}
\end{aligned}
$$

where $\hat{\mathbf{n}} \equiv \hat{\mathbf{n}}_{\perp\mathrm{above}}$ is the unit normal vector <u>above</u> the interface.

Boundary Conditions on $\mathbf{E}$ (contd.)

- The two previous results can be combined as:

$$
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E^{\perp}_{\text{above}}(\mathbf{r}) - E^{\perp}_{\text{below}}(\mathbf{r}) &= \frac{\sigma(\mathbf{r})}{\epsilon_0} \\
\mathrm{E}^{||}_{\text{above}}(\mathbf{r}) &= \mathrm{E}^{||}_{\text{below}}(\mathbf{r}) \\
\implies \mathbf{E}_{\text{above}}(\mathbf{r}) - \mathbf{E}_{\text{below}}(\mathbf{r}) &= \frac{\sigma(\mathbf{r})}{\epsilon_0}\hat{\mathbf{n}}
\end{aligned}
$$

where $\hat{\mathbf{n}} \equiv \hat{\mathbf{n}}_{\perp\text{above}}$ is the unit normal vector above the interface.

- Alternatively taking dot products with $\hat{\mathbf{n}}$ the combined result is:

$$\nabla V_{\text{above}}(\mathbf{r}) \cdot \hat{\mathbf{n}} - \nabla V_{\text{below}}(\mathbf{r}) \cdot \hat{\mathbf{n}} = -\frac{\sigma(\mathbf{r})}{\epsilon_0}.$$

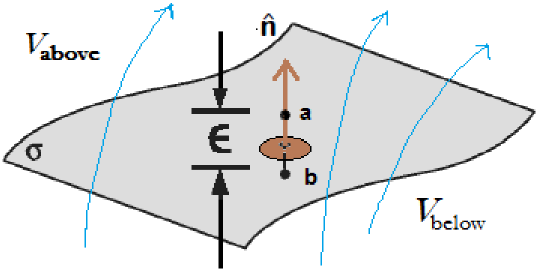
- Introducing **Normal directional derivative:** $D_{\hat{\mathbf{n}}} V \equiv \frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}$

$$\frac{\partial V_{\text{above}}(\mathbf{r})}{\partial n} - \frac{\partial V_{\text{below}}(\mathbf{r})}{\partial n} = -\frac{\sigma(\mathbf{r})}{\epsilon_0}.$$

Boundary Condition on V

- Consider points **a** and **b** just above and below the interface separated by an infinitesimal amount $\epsilon \to 0$. The potential difference is given by

$$\lim_{\epsilon \to 0} [V_{\mathrm{above}}(\mathbf{a}) - V_{\mathrm{below}}(\mathbf{b})] = \lim_{\mathbf{a} \to \mathbf{b}} \left[\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{r} \right] = 0$$



$$V_{\mathrm{above}} = V_{\mathrm{below}}$$

Boundary Conditions: Summary

- Boundary conditions on $\mathbf{E}$ and V apply to all types of **smooth** surfaces, **flat or curved**, or whether they happen to be **charged or not**
- Based on 2 basic principles: (1) Gauss's law and (2) Conservative nature of $\mathbf{E}$.

Fact

1. The normal component of the Electric field, $\mathbf{E}_\perp$ is discontinuous by an amount σ/ϵ_0 across any boundary.
2. The tangential component of Electric field, $\mathbf{E}_{||}$ is continuous across any boundary.
3. The Electrostatic potential V is continuous across any boundary.
4. The **Normal derivative** of the potential, $\frac{\partial V}{\partial n}$ is discontinuous by an amount σ/ϵ_0 across any boundary.