Physics II (PH 102)

Electromagnetism (Lecture 8 & 9)

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Feb 2020

Electrostatic Potential Energy of a Point Test Charge in Electric Field

Consider a stationary configuration of source charge distribution.

- Let $\mathbf{E}(\mathbf{r})$ be a pre-existing electric field with potential $V(\mathbf{r})$ at a point P.
- The electrostatic force $\mathbf{F}_{field}$ on a positive test charge Q is

$$\mathbf{F}_{field} = Q\mathbf{E} = -\mathbf{F}_{ext}.$$

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Definition

Electrostatic Potential Energy of a test charge Q at point P($\mathbf{r}$) *is equal to the amount of work done by an external agent against the electrostatic field to bring the charge Q from ∞ (or* **ref***) to the point $P(\mathbf{r})$:*

$$U_E(\mathbf{r}) = \int_{\infty}^{\mathbf{r}} \mathbf{F}_{ext} \cdot d\mathbf{r}' = -\int_{\infty}^{\mathbf{r}} \mathbf{F}_{field} \cdot d\mathbf{r}' = -Q\int_{\infty}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{r}' = QV(\mathbf{r}).$$

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- *Electrostatic Energy difference* between two points **a** and **b** can be unambiguously expressed in terms of the *Potential difference*:

$$U_E(\mathbf{b}) - U_E(\mathbf{a}) = Q[V(\mathbf{b}) - V(\mathbf{a})] = -Q\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}'.$$

Potential Energy due to a system of Point Charges in free Field

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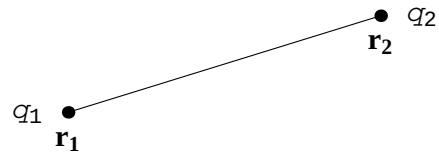

$$W_2 = W_1 + \delta w_2 = 0 + \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}$$

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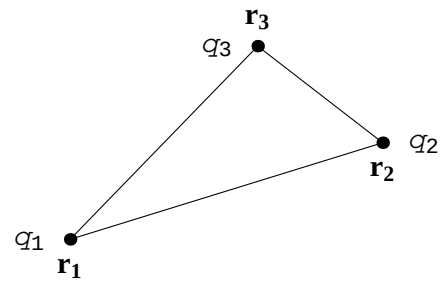

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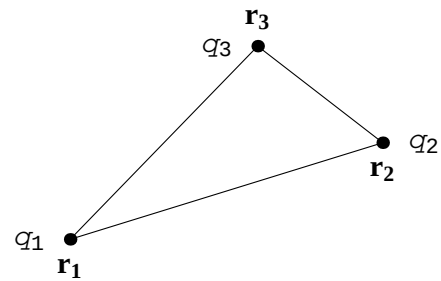

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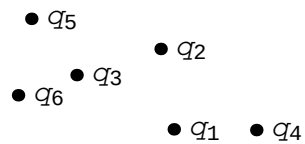

- Generalize formula up to n charges:
  ($j \neq i$ Self interactions excluded!)

$$W_n = W_{n-1} + \delta w_n = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n-1}\sum_{j=2,\, j>i}^{n}\left(\frac{q_i q_j}{r_{ij}}\right) \rightarrow \text{No Double Counting}$$

$$\equiv \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=1,\, j\neq i}^{n}\left(\frac{q_i q_j}{r_{ij}}\right) \rightarrow \text{With Double Counting}$$

Potential Energy of a system of Point Charges in free Field (contd.)

- The Electrostatic Potential Energy is equal to the total work done W_n to assemble the configuration of n point charges at $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n$:

$$U_E(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n) \equiv W_n = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \left(\frac{q_i q_j}{r_{ij}} \right)$$

$$= \frac{1}{2} \sum_{i=1}^{n} q_i \left(\sum_{j=1, j\neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}} \right)$$

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- $V_i \equiv V(\mathbf{r}_i)$: Potential at the i^{th} charge site $\mathbf{r}_i$ due to other $n-1$ charges.

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- The result is independent of the order/sequence in which the charges are assembled at the respective EXACT locations, $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n$.

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- The result OBVIOUSLY, depends on the respective locations of the charges. Thus, U_E is called the CONFIGURATION ENERGY.

Configuration Energy of Point Charges in free space

- Superposition Principle is invalid (1/2 factor avoids **double counting**!):

$$U_E = \frac{1}{2}\sum_{i=1}^{n} u_{Ei}$$

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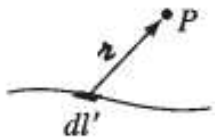
- SELF-ENERGIES of the individual point charges were excluded!

Definition

SELF-ENERGY: This is the amount of energy needed to fabricate or build-up the individual point charges by bringing their respective differential amounts of constituent charges from ∞ *to the specific locations.*

Electrostatic Potential Energy of General Distribution of Source Charges

(a) Discrete charges



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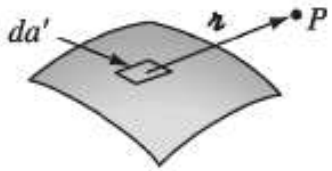

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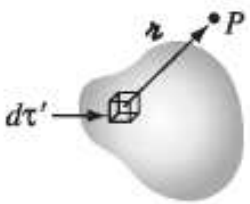

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Localized charge distributions:

- Volume $\mathcal{V}$ with volume charge density $\rho(\mathbf{r})$
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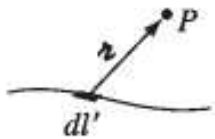

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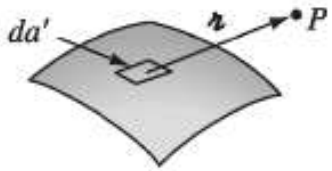

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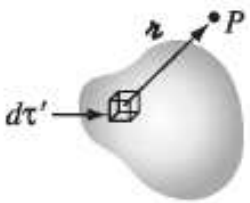

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- What else is missing here?

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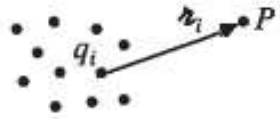

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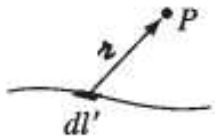

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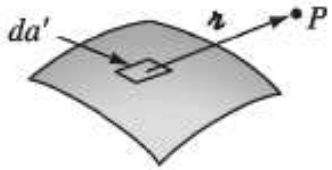

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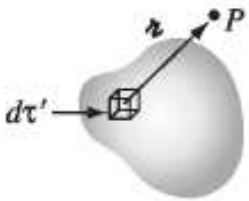

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  INTERACTION ENERGIES
- Superposition Principle is invalid in general!

True Configuration Energy of General Localized Charge Distribution

- Consider the most general localized charge distribution $\rho_{\rm tot}$ in a region $\mathcal{V}$, bounded by a closed surface S :

$$U_E = \frac{1}{2} \iiint_{\mathcal{V}} \rho_{\rm tot}(\mathbf{r}')\, V(\mathbf{r}')\, d\tau'.$$

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- Using Gauss's differential law at source: $\rho_{\text{tot}}(\mathbf{r}') = \epsilon_0 \nabla' \cdot \mathbf{E}(\mathbf{r}')$,

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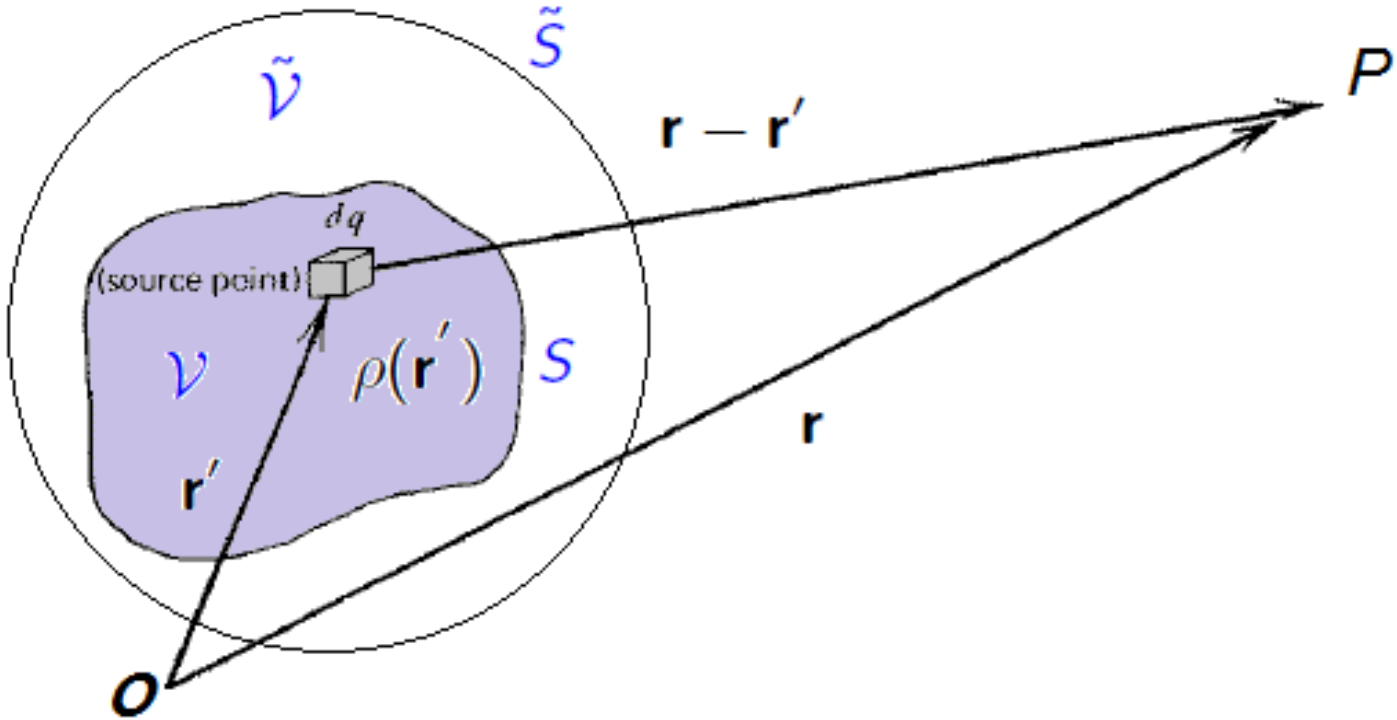
- In the last step, we applied the Gauss' Divergence Theorem to obtain the surface integral over the bounding surface S.

Recall Identity: $\nabla \cdot (V\,\mathbf{E}) = V\,(\nabla \cdot \mathbf{E}) + (\nabla V) \cdot \mathbf{E} = V\,(\nabla \cdot \mathbf{E}) - |\mathbf{E}|^2$

- Let us EXTEND the integration over a very large SPHERICAL volume $\tilde{\mathcal{V}}$, with bounding surface $\tilde{S}$, enclosing the LOCALIZED distribution $\mathcal{V}$:

$$U_E = \frac{1}{2} \iiint_{\mathcal{V}} \rho_{\mathrm{tot}}(\mathbf{r}')V(\mathbf{r}')\,d\tau' = \frac{1}{2} \iiint_{\tilde{\mathcal{V}}} \rho_{\mathrm{tot}}(\mathbf{r}')V(\mathbf{r}')\,d\tau'$$



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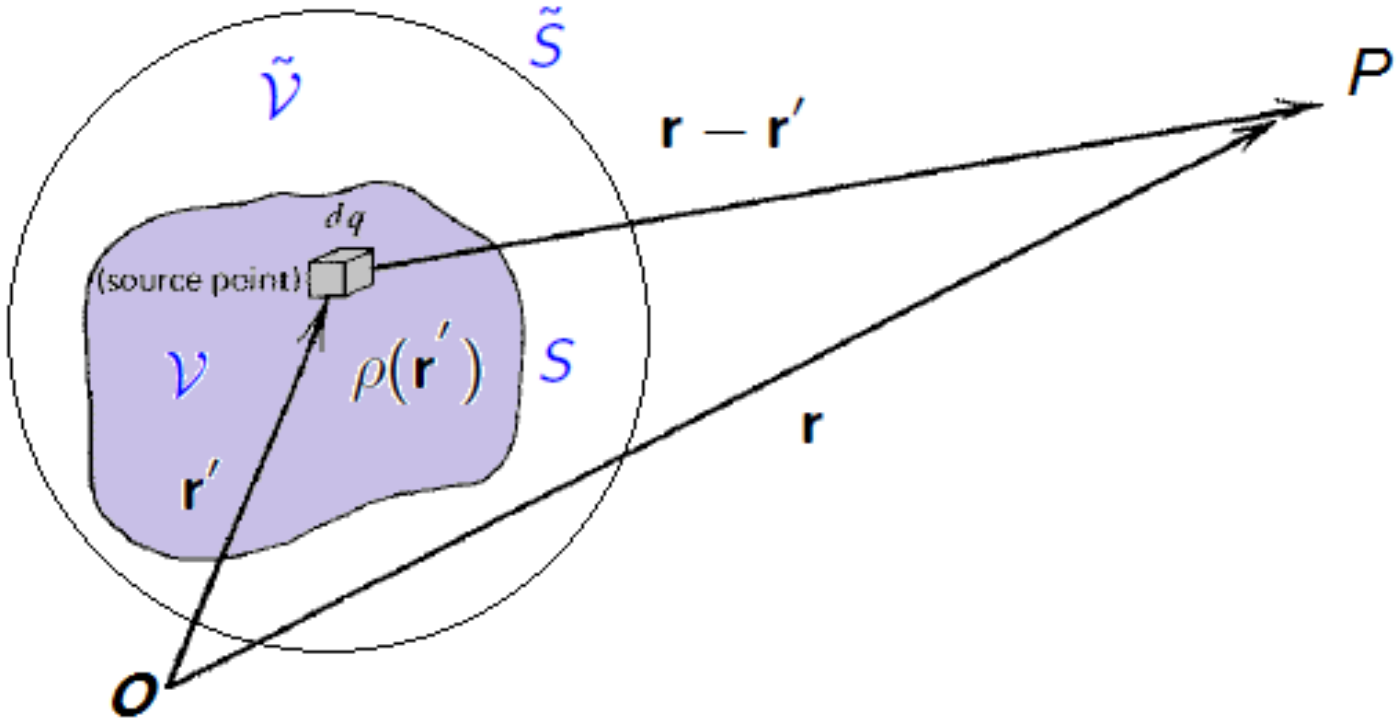

- For <u>localized</u> distribution at a distant point P, $V(\mathbf{r}) \propto 1/r$, $|\mathbf{E}(\mathbf{r})| \propto 1/r^2$
- For points $\mathbf{r}'$ on this very large spherical surface $\tilde{S}$,

$$\oiint_{\tilde{S}} [V(\mathbf{r}')\,\mathbf{E}(\mathbf{r}')]_{\rm boundary}\cdot d\mathbf{a}' \propto \left(\frac{1}{r'}\cdot\frac{1}{r'^2}\cdot r'^2\right)_{\tilde{S}} \sim \#\left(\frac{1}{r'}\right)_{\tilde{S}} \to 0,\ r' \to \infty.$$

True Configuration Energy of General Charge Distribution (contd.)

- Extending to include **ALL SPACE**: $\tilde{\mathcal{V}} \to \mathcal{V}_\infty \equiv \mathbb{R}^3$ and $\tilde{S} \to S_\infty$, then the surface integral vanishes!

$$
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True Configuration Energy of General Charge Distribution (contd.)

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In Summary: 2 Prescriptions to determine the Configuration Energy of a Localized distribution (depending on convenience) in a given problem

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Example

Determine the total configuration/self energy of a uniformly charged solid sphere V_R of radius R and charge q.

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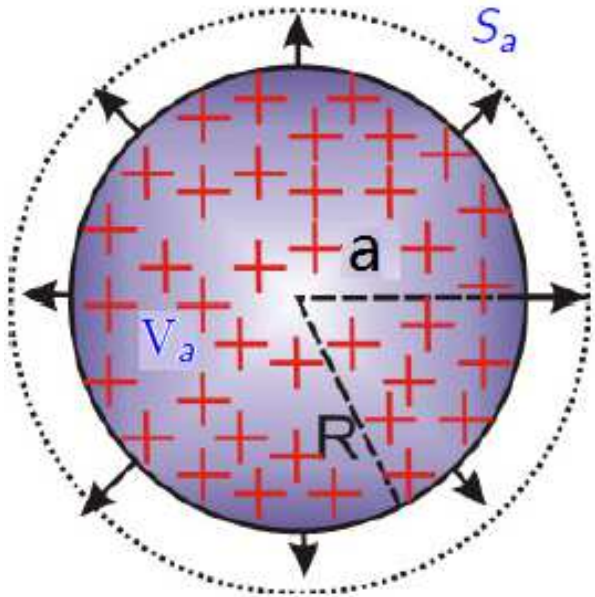

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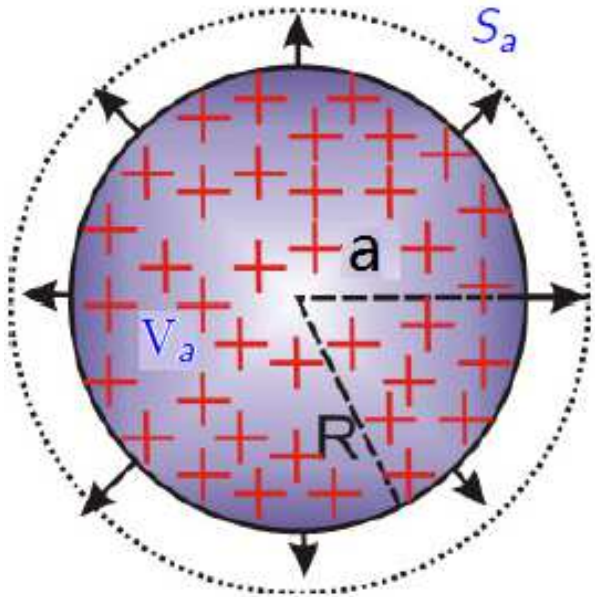

$$U_E = \frac{1}{2} \iiint_{V_R \to V_a} \rho(\mathbf{r})\, V(\mathbf{r})\, d\tau = \frac{\epsilon_0}{2} \iiint_{V_a} |\mathbf{E}(\mathbf{r})|^2 \, d\tau + \frac{\epsilon_0}{2} \oiint_{S_a} V(\mathbf{r})\, \mathbf{E}(\mathbf{r}) \cdot d\mathbf{a}$$

- Electric fields due to **original** charged sphere V_R (by Using Gauss's Law):

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{\mathbf{r}} & r < R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} & r > R \end{cases}$$

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- Since the radius δR of the point charge vanishes, the self-energy blow up!
- Point charges are idealized concepts. In reality $\delta R \neq 0$ (say, for electrons), so **Self-Energies** (classically) of spherical objects of finite radius δR is:

$$W_{\text{sphere}} = \frac{1}{4\pi\epsilon_0}\left(\frac{3q^2}{5\,\delta R}\right) \rightarrow \text{finite}$$

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Find the Interaction energy of two charges, q_1 and q_2 located at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively.

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$\Rightarrow$ This is exactly the work done by an external agent in bringing q_2 from ∞ to $\mathbf{r}_2$ with q_1 already present at $\mathbf{r}_1$.

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- It is conventional to define an ENERGY DENSITY:

$$u(\mathbf{r}) = \frac{\epsilon_0}{2} |\mathbf{E}(\mathbf{r})|^2$$

$\Longmapsto$ a volume dv will contain Electrostatic Potential Energy equal to $u(\mathbf{r})dv$.