Physics II (PH 102)

Electromagnetism (Lecture 11)

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Method of Images: Avoids solving PDEs in Boundary Valued Problems

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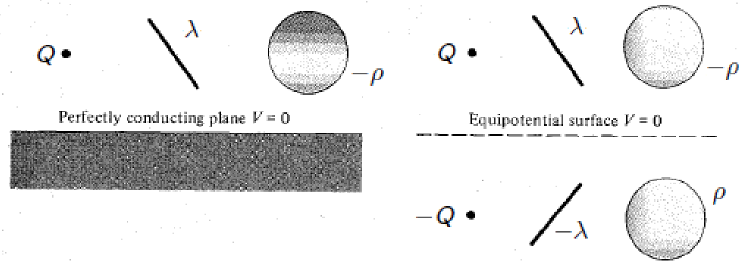
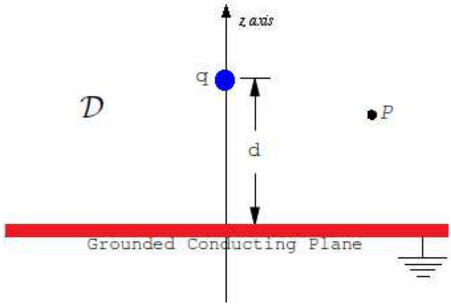
1. Central idea: Map the original hard problem to another easier problem, *but satisfying the same boundary conditions*. Then *Uniqueness Theorem* guarantees the correctness of the solution.
2. Use Fact: All conducting surfaces are represented by equipotentials.
3. Strategy: All Real charge configurations and conducting surfaces are replaced by the same Real charges, equipotential surfaces and some additional **fictitious** charges or charge distributions in the conducting region, called *Image Charges*.



The Classic Image Problem

Example

Suppose a point charge q is held at a distance d above a infinite grounded conducting plane. What is the Electrostatic Potential at point P in the non-conducting region $\mathcal{D}$ above the conducting plane?



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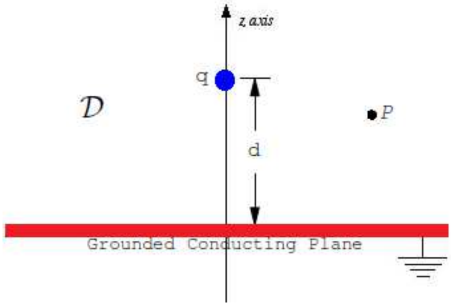

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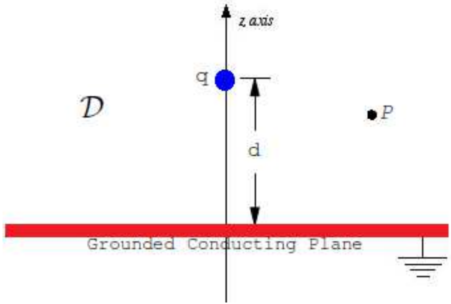
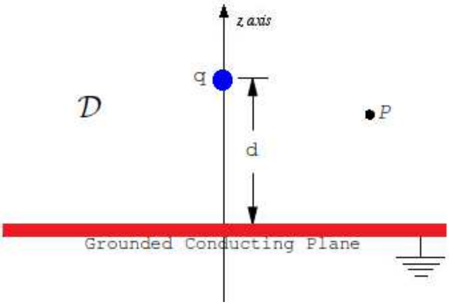


- The Electrostatic Potential V at point $P \equiv (x, y, z)$ will be due to point charge q and the induced surface charges.
- Problem is, we do not know $\sigma(x, y)$ *a priori* ! How to determine $V(x, y, z)$ without directly knowing $\sigma(x, y)$ on the conducting plane?

Infinite grounded conducting plane



Set up a co-ordinate system with xy-plane as the given infinite conducting plane and q lies on the z-axis:

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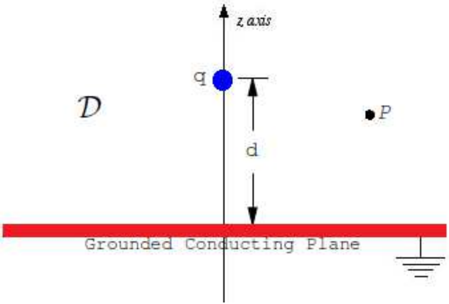

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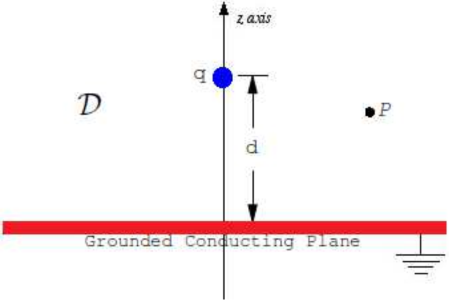

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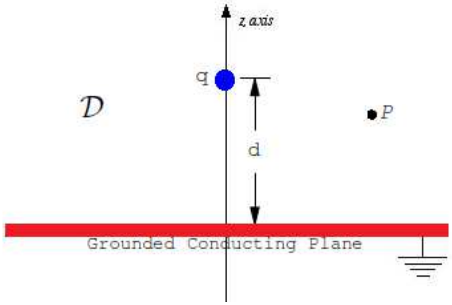

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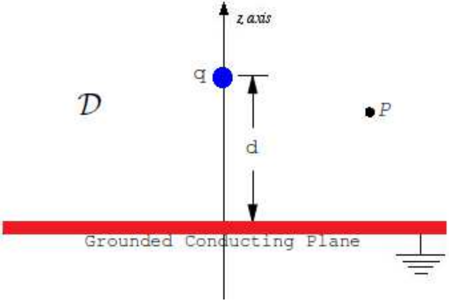

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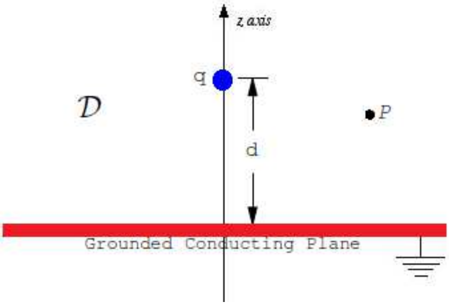

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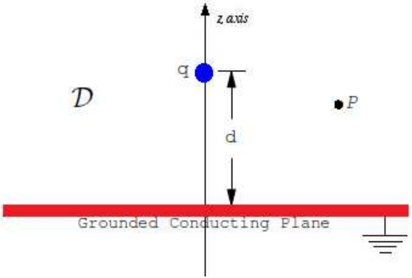
- **Boundary Condition** on V in the original problem:

$$V(\mathbf{S}) = 0, \quad \forall \mathbf{S} \in S.$$

Infinite grounded conducting plane (contd.)

Real System is mapped on to the Fictitious System satisfying the same b.c.



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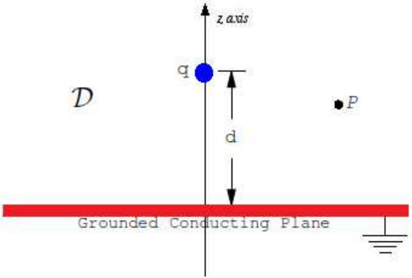

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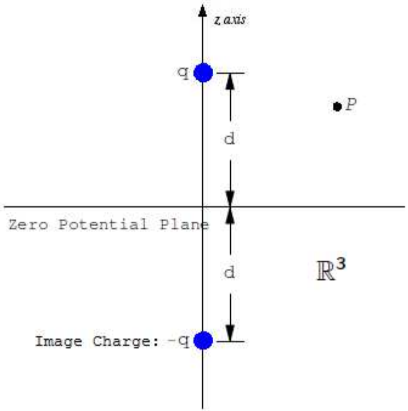

Potential $V'(x, y, z)$ defined in the whole of $\mathbb{R}^3$

The Fictitious System

Infinite grounded conducting plane (contd.)

Consider the **Fictitious System**:

- Charge distribution: $\rho'(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}_0) + (-q)\delta^3(\mathbf{r} + \mathbf{r}_0); \quad \mathbf{r}_0 = (0, 0, d)$

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1. Real charge configuration in the common region $\mathcal{D}$ is identical:

$$\rho'(\mathbf{r})|_{\mathcal{D}} \xrightarrow{z>0} \rho(\mathbf{r})$$

2. Boundary conditions in the common region $\mathcal{D}$ are identical:

$$\nabla^2 V' = \frac{1}{\epsilon_0}\rho' \quad \text{over } \mathbb{R}^3 \quad \xrightarrow{z>0} \quad \nabla^2 V = \frac{1}{\epsilon_0}\rho \quad \text{over } \mathcal{D} \subset \mathbb{R}^3,$$

$$V' = 0 \quad \text{on } S' \quad \xrightarrow{z>0} \quad V = 0 \quad \text{on } S \subset S'$$

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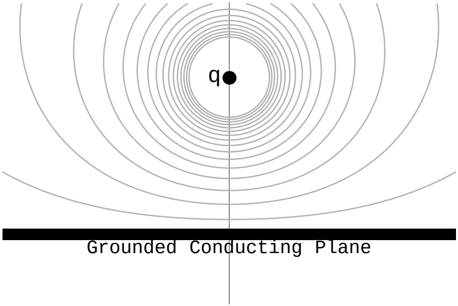

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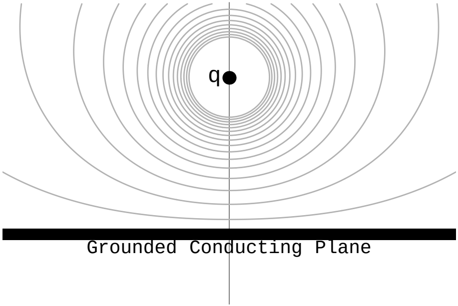
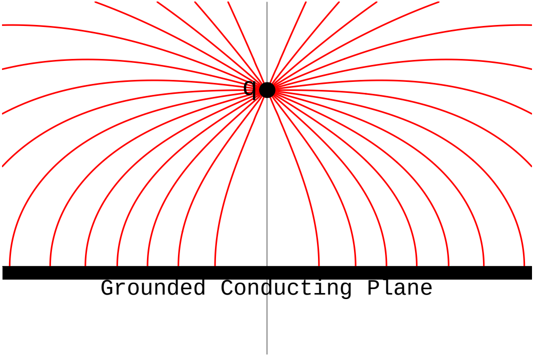


Note: We *needn't bother* that $V = V'$ yields wrong result for $z < 0$!

Infinite grounded conducting plane: Electric Field

Electrostatic Field in $\mathcal{D}$: $\mathbf{E}(x,y,z) = \mathbf{E}'(x,y,z \geq 0) = -\nabla V(x,y,z)$, i.e.,

$$\mathbf{E}(x,y,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z-d)\hat{\mathbf{k}}}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z+d)\hat{\mathbf{k}}}{(x^2+y^2+(z+d)^2)^{3/2}}\right]$$



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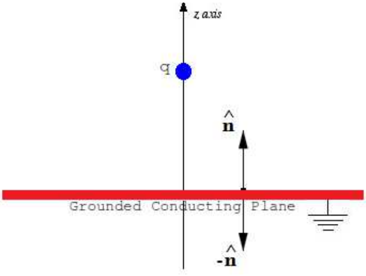
Infinite grounded conducting plane: Surface Charge Density



- Normal to conductor: $\pm\hat{\mathbf{n}} = \pm\hat{\mathbf{k}}$
- Induced surface charge density $E_{\perp} = \sigma/\epsilon_0$:

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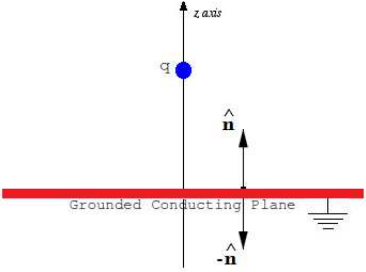

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$$\equiv \epsilon_0 \mathbf{E}(x,y,z=0)\cdot\hat{\mathbf{k}}$$

$$\sigma(x,y) = -\frac{q}{2\pi}\frac{d}{(x^2+y^2+d^2)^{3/2}}$$

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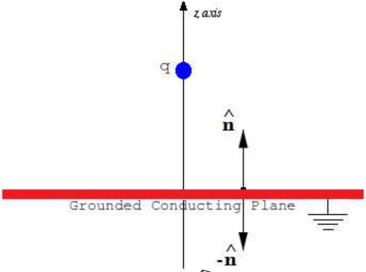

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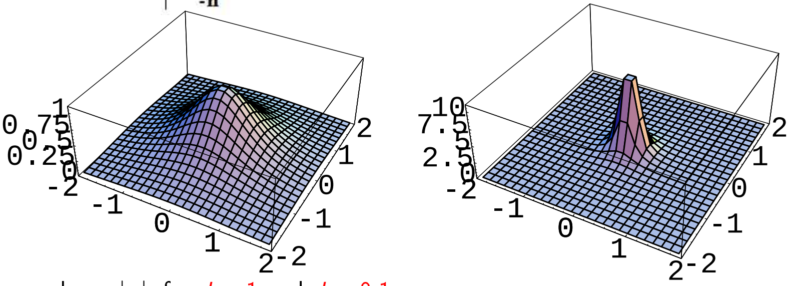

Figure shows $|\sigma|$, for $d = 1$ and $d = 0.1$

$\Longrightarrow$ Maximum induced charge density is right below the point charge

Infinite grounded conducting plane: Total Induced Charge

- Total induced surface charge: (with $dA = dx\, dy = s\, ds\, d\phi$)

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&= \quad -\frac{qd}{2\pi} \left[-\frac{1}{\left(s^2+d^2\right)^{1/2}} \right]_{s=0}^{s=\infty} (2\pi) \\
Q_{\text{induced}} &= \quad -q
\end{aligned}
$$

Infinite grounded conducting plane: Force on Real Charge q

Electric Field at $P(x, y, z)$ in $\mathcal{D}$, that we have already calculated:

$$\mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[q\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z-d)\hat{\mathbf{k}}}{(x^2+y^2+(z-d)^2)^{3/2}}\right) + (-q)\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z+d)\hat{\mathbf{k}}}{(x^2+y^2+(z+d)^2)^{3/2}}\right)\right]$$

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- Electric Field $\rightarrow$ *Induced charges* $\equiv$ Electric Field $\rightarrow$ *Image charge*:

$$\mathbf{E}_{\text{induced}}(x,y,z) \equiv \mathbf{E}_{\text{image}}(x,y,z) = \frac{(-q)}{4\pi\epsilon_0}\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+(z+d)\hat{\mathbf{k}}}{(x^2+y^2+(z+d)^2)^{3/2}}$$

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- Force on q due to conducting plane $\equiv$ Force on q due to Image charge:

$$\begin{aligned}\mathbf{F}_q &= q\mathbf{E}_{\text{induced}}(0,0,d) \equiv q\mathbf{E}_{\text{image}}(0,0,d) \\ &= -\frac{q^2\hat{\mathbf{z}}}{4\pi\epsilon_0(2d)^2} \rightarrow \text{Attractive force}\end{aligned}$$

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$$\begin{aligned}\mathbf{F}_q &= q\mathbf{E}_{\text{induced}}(0,0,d) \equiv q\mathbf{E}_{\text{image}}(0,0,d)\\ &= -\frac{q^2\hat{\mathbf{z}}}{4\pi\epsilon_0(2d)^2} \rightarrow \text{Attractive force}\end{aligned}$$

***QUESTION:** Is there any difference in calculated physical quantities in non-conducting region $\mathcal{D}$, between those obtained from the *Fictitious System* (charge-image) and those from the *Real System* (charge-conductor)?

Infinite grounded conducting plane: Electrostatic Energy??

- Configuration energy of *Real System:*
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- Intuitive way of understanding this difference is to use the **integral formula**:

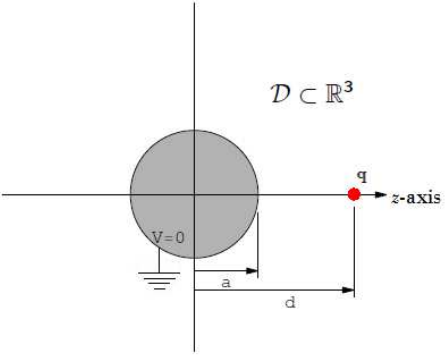
$$U_E(\text{Fictitious}) = \frac{\epsilon_0}{2}\iiint_{\mathbb{R}^3} E^2\,dv = 2\cdot\frac{\epsilon_0}{2}\iiint_{\mathcal{D}} E^2\,dv = 2U_E(\text{Real}).$$

$\Longrightarrow$ *The true domain of integration $\mathcal{D}$ is only half the domain $\mathbb{R}^3$ for the Fictitious System.*

Another classic image problem: Grounded Conducting Sphere

Example

Consider a grounded conducting sphere of radius a and a charge q held at a distance of d from the center. What is the potential in region $\mathcal{D}$ outside the conducting sphere?



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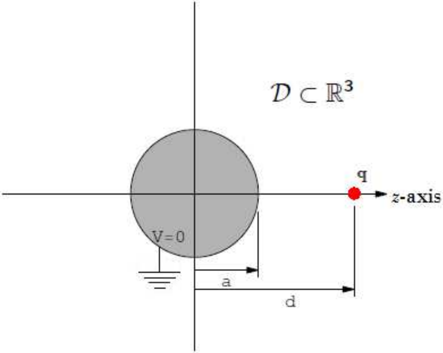

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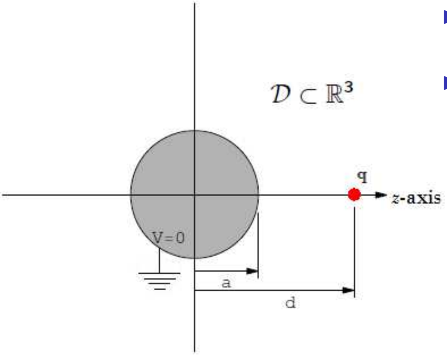

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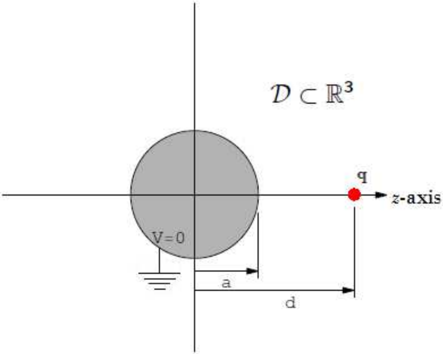

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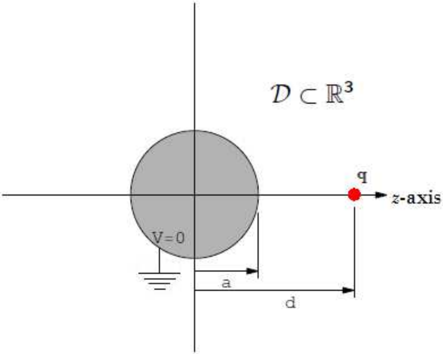

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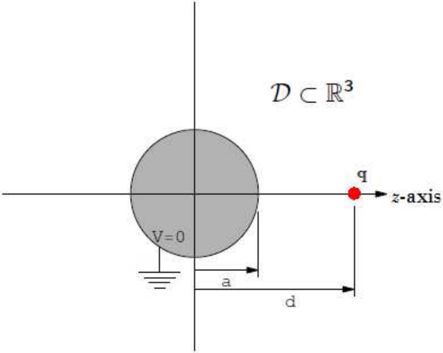

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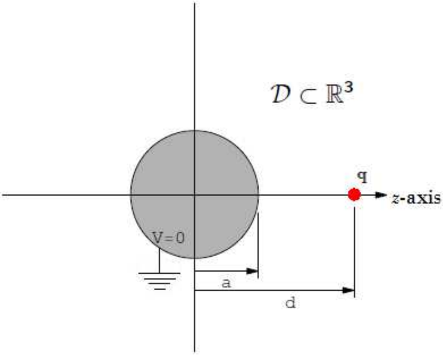

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$$V(\mathbf{S}) = 0, \;\; \forall \mathbf{S} \in S$$

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Replace Real System with Fictitious System: *Real charge q, Image charge q′ & Equipotential surface* $\Longrightarrow$ $V'(\mathbf{r})$ in $\mathbb{R}^3$ is identical to $V(\mathbf{r})$ in $\mathcal{D}$.

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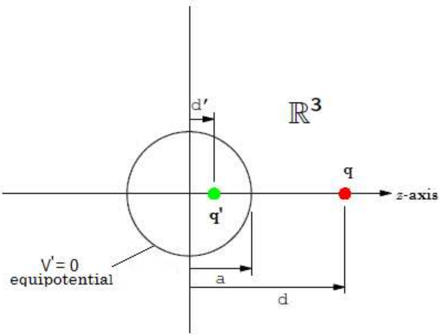

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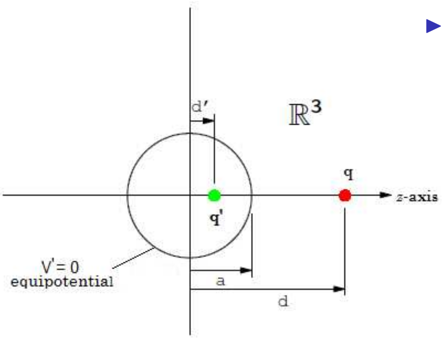

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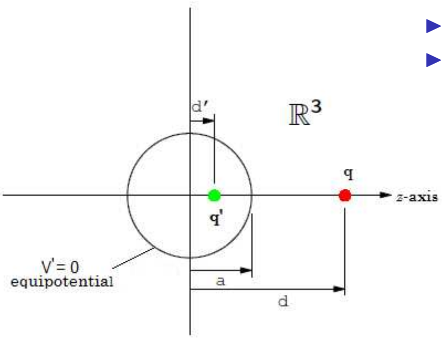

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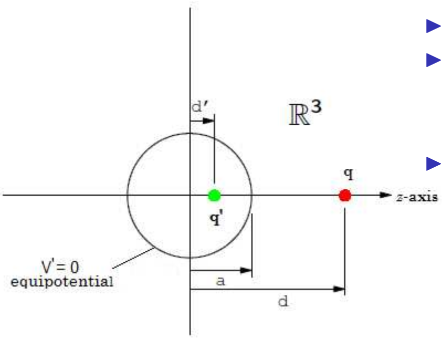

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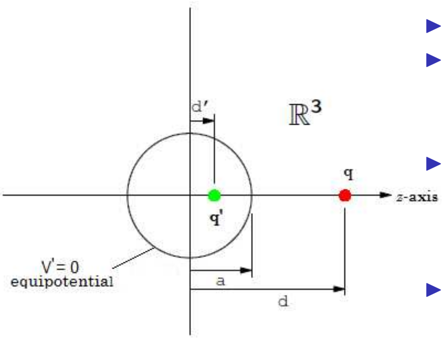

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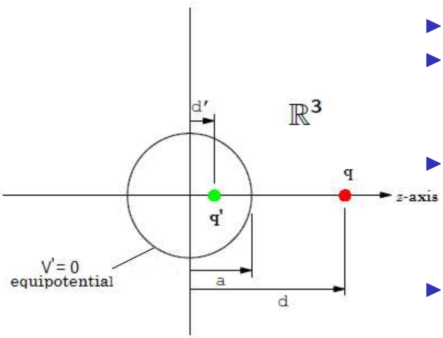

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If such a q' and d' can be found, then we have nailed the problem!