Physics II (PH 102)

Electromagnetism (Lecture 12)

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Method of Separation of Variables

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$$V(x, y, z) = X(x) + Y(y) + Z(z) \qquad \text{or} \qquad V(x, y, z) = X(x)Y(y)Z(z).$$

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  - *This method does not ensure the most general solutions, but rather yields a sub-class of all possible solutions that are separable.*
- *Uniqueness Theorem*: For given **Set of Boundary Conditions** it guarantees the correct answer irrespective to type to ansatz or methodology.
- *Linearity* property of Laplace's solution: If $V_1, V_2, V_3, \cdots$ satisfy Laplace's equation, so does any linear combinations of them, i.e., if

  $$V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \cdots$$

  where α_i are arbitrary <u>real</u> constants, then

  $$\nabla^2 V = \alpha_1 \cancelto{0}{\nabla^2 V_1} + \alpha_2 \cancelto{0}{\nabla^2 V_2} + \alpha_3 \cancelto{0}{\nabla^2 V_3} + \cdots = 0.$$

Solution via Additive Ansatz

Solve the 2D Laplace's Equation in Cartesian co-ordinates:

$$\nabla^2 V(x,y) = \frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0$$

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$$\begin{aligned} X(x) &= \frac{1}{2}\alpha x^2 + \beta x + \delta \\ Y(y) &= -\frac{1}{2}\alpha y^2 + \gamma y + \rho \end{aligned}$$

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This Additive Ansatz yields problematic unphysical solutions for potentials due to localized charge distributions, as they do not die away as $x, y \rightarrow \pm\infty$!

Solution via Multiplicative Ansatz

- Try ***Product ansatz***: $V(x,y) = X(x)Y(y)$; $X, Y \to$ smooth functions.

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$\implies$ Single 2^{nd}order PDE gets reduced to two 2^{nd} ODEs. The choice $\pm$ is dictated by the specific nature of the problem and b.c.

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- E.g., $+k^2$ choice leads to the full solution as combinations of **oscillatory & exponential** functions,

$$V(x,y) = X(x)Y(y) = (A\cos kx + B\sin kx)\underbrace{(C\cosh ky + D\sinh ky)}_{const.e^{-ky}},$$

$\implies$ yields correct physical nature of potentials due to localized distributions.

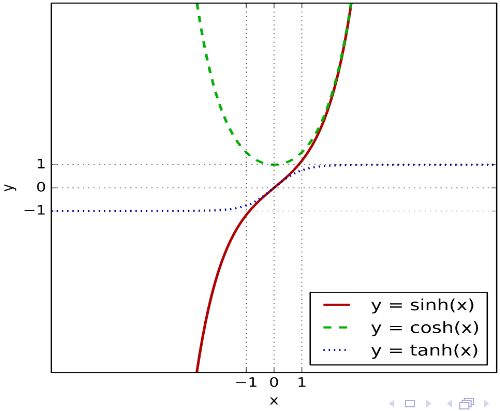
Hyperbolic Functions

They are analogs of of ordinary trigonometrical functions:

$$\cosh x = \frac{\exp(x) + \exp(-x)}{2}$$

$$\sinh x = \frac{\exp(x) - \exp(-x)}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\exp(2x) - 1}{\exp(2x) + 1}$$



Properties of Solutions obtained via Variable Separable Ansatz

There exists a *complete* and *orthonormal* set of *basis functions S* for expansion of any function, say, $X(x)$, obtained as a solution to the Laplace's equation via the separation of variables ansatz:

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- **Completeness:** If the solution function $X(x)$ defined over the given domain, $x \in \mathbb{D}[a, b] \subset \mathbb{R}$, can be expanded as arbitrary *linear combination* of so-called "*basis functions*" $f_n(x)$:

$$X(x) = \sum_{n=0}^{\infty} C_n \, f_n(x) \; ; \quad C_n \in \mathbb{R} \quad \& \quad f_n(x) \in S.$$

Fact: The Basis Functions $f_n(x) \in S$ defined on domain $\mathbb{D}[a, b] \subset \mathbb{R}$ span an ∞-dimensional vector space of solutions, $F = \{X(x) \,|\, x \in \mathbb{D}[a, b] \subset \mathbb{R}\}$, termed as a FUNCTION SPACE, where,

$$S = \{f_n(x) \,|\, n \in \mathbb{Z}, \; x \in \mathbb{D}[a, b] \subset \mathbb{R}\}$$

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- **Orthonormality of Basis:** If the set of functions, $f_n(x) \in S$ defined on the domain $\mathbb{D}[a, b] \subset \mathbb{R}$ is such that their convolution :

$$\int_a^b f_n(x) \, f_m(x) \, dx = const. \, \delta_{nm} = \begin{cases} const. & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases}$$

Basis Set of a Function Space of Laplace's Solutions

Example

Sine and Cosine functions can form a *complete* and *orthonormal* basis set S in a certain domain, say, $x \in \mathbb{D}[\gamma, \gamma + 2l] \subset \mathbb{R}$:

$$S = \left\{ \sin\left(\frac{n\pi x}{l}\right), \cos\left(\frac{n\pi x}{l}\right) \mid n \in \mathbb{Z},\, x \in \mathbb{D}[\gamma, \gamma + 2l] \subset \mathbb{R} \right\}$$

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- **Orthonormality**:(Take e.g., $\gamma = 0,\ 2l = 2\pi$)

$$\begin{aligned}
\int_{\gamma}^{\gamma+2l} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx &= l\delta_{nm}, \\
\int_{\gamma}^{\gamma+2l} \cos\left(\frac{n\pi x}{l}\right) \cos\left(\frac{m\pi x}{l}\right) dx &= l\delta_{nm}, \\
\int_{\gamma}^{\gamma+2l} \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{m\pi x}{l}\right) dx &= 0, \qquad \text{where } m, n \in \mathbb{Z}.
\end{aligned}$$

- **Completeness**: Any arbitrary Harmonic Function can be expanded in an infinite series of Sines and Cosines basis functions. Such a series is termed as a *Fourier Expansion*.

Fourier Series: Topic of Harmonic Analysis

The Fourier Expansion is valid for all Harmonic Functions $f(x)$ because they are *Piecewise Regular* in a given domain $\mathbb{D}$, i.e.,

- $f(x)$ must be single valued in $\mathbb{D}$.
- $f(x)$ can atmost have finite number of finite discontinuities in $\mathbb{D}$.
- $f(x)$ must have finite number of minima or maxima in $\mathbb{D}$.

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A *Fourier Expansion* is defined as an expansion of a *Piecewise Regular* function, say $f(x)$, defined over a *Principal domain* $\mathbb{D} \equiv [\gamma \leq x \leq (\gamma + 2l)] \in \mathbb{R}$ and having *Period* $T = 2l$ outside this interval $\mathbb{D}$, in an infinite series of sine and cosine functions:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[a_m \cos\left(\frac{m\pi x}{l}\right) + b_m \sin\left(\frac{m\pi x}{l}\right) \right].$$

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The real coefficients of this series are called *Fourier Coefficients*:

$$a_n = \frac{1}{l} \int_{\gamma}^{\gamma+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx \quad ; \quad n \geq 0$$

$$b_n = \frac{1}{l} \int_{\gamma}^{\gamma+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx \quad ; \quad n \geq 1.$$

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Finally, simply integrating both sides over $\mathbb{D}[\gamma, \gamma + 2l]$

$$\int_{\gamma}^{\gamma+2l} f(x) dx = \frac{a_0}{2} \int_{\gamma}^{\gamma+2l} dx + \sum_{m=1}^{\infty} \int_{\gamma}^{\gamma+2l} \left[\cancelto{0}{a_m \cos\left(\frac{m\pi x}{l}\right)} + \cancelto{0}{b_m \sin\left(\frac{m\pi x}{l}\right)} \right] dx$$

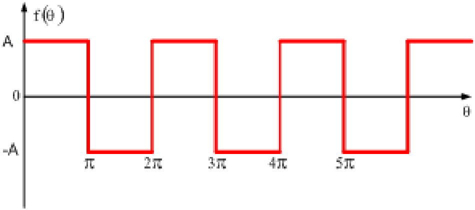
$$\int_{\gamma}^{\gamma+2l} f(x) dx = l a_0$$

$$a_0 = \frac{1}{l} \int_{\gamma}^{\gamma+2l} f(x)\, dx$$

Fourier Harmonic Analysis

Example

Find the Fourier series of the following periodic function
(*Square Pulse*)



$$f(\theta) = A \quad \text{when} \quad 0 < \theta < \pi$$
$$= -A \quad \text{when} \quad \pi < \theta < 2\pi$$

$$f(\theta + 2\pi) = f(\theta)$$

Does the function satisfy DIRICHLET's conditions to be Fourier Expanded?

Fourier Coefficients: Here, $\mathbb{D}[\gamma = 0,\ \gamma + 2l = 2\pi]$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta$$
$$= \frac{1}{\pi}\left[\int_0^{\pi} f(\theta)\,d\theta + \int_{\pi}^{2\pi} f(\theta)\,d\theta\right]$$
$$= \frac{1}{\pi}\left[\int_0^{\pi} A\,d\theta + \int_{\pi}^{2\pi} -A\,d\theta\right]$$
$$= 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta$$
$$= \frac{1}{\pi}\left[\int_0^{\pi} A\cos n\theta\,d\theta + \int_{\pi}^{2\pi}(-A)\cos n\theta\,d\theta\right]$$
$$= \frac{1}{\pi}\left[A\frac{\sin n\theta}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[-A\frac{\sin n\theta}{n}\right]_{\pi}^{2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta$$
$$= \frac{1}{\pi}\left[\int_0^{\pi} A\sin n\theta\,d\theta + \int_{\pi}^{2\pi}(-A)\sin n\theta\,d\theta\right]$$
$$= \frac{1}{\pi}\left[-A\frac{\cos n\theta}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[A\frac{\cos n\theta}{n}\right]_{\pi}^{2\pi}$$
$$= \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

Fourier Coefficients

$$b_n = \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

$$= \frac{A}{n\pi}\left[1+1+1+1\right]$$

$$\boxed{b_n = \frac{4A}{n\pi} \quad \text{when n is odd}}$$

$$b_n = \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

$$= \frac{A}{n\pi}\left[-1+1+1-1\right]$$

$$\boxed{b_n = 0 \quad \text{when n is even}}$$

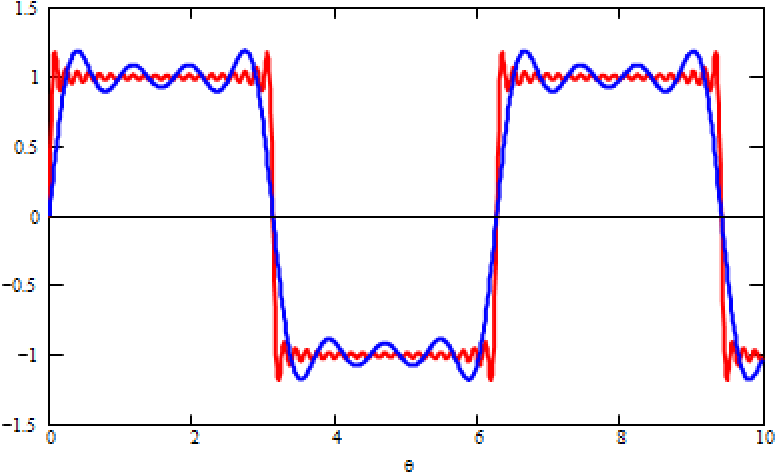
Therefore, the corresponding Fourier series is

$$f(\theta) = \frac{4A}{\pi}\left(\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \frac{1}{7}\sin 7\theta + \cdots\right)$$

In writing the Fourier series it is not possible to consider infinite number of terms (HARMONICS) for practical reasons. The question therefore, is – how many harmonics do we consider?

$$f(\theta) = \begin{cases} A & \text{when} \quad 0 < \theta < \pi \\ -A & \text{when} \quad \pi < \theta < 2\pi \end{cases} = \frac{4A}{\pi}\sum_{n=1}^{n=\infty}\frac{1}{2n-1}\sin\left[(2n-1)\theta\right]$$

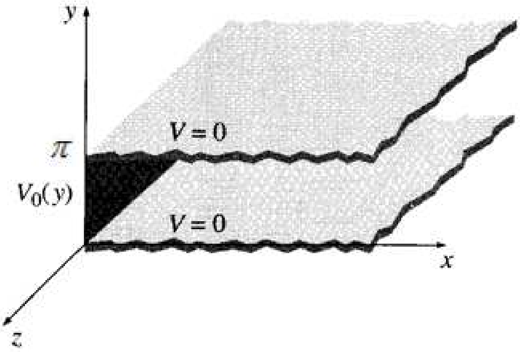
The red curve was drawn with 20 harmonics and the blue curve was drawn with 4 harmonics.



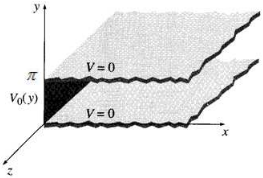
2D Boundary Valued Problem in Cartesian System

Example

Two infinite grounded metal plates lie parallel to the *xz*-planes, one at $y = 0$ and the other at $y = \pi$. The left end is closed off with an infinite strip insulated from the two plates and maintained at a specific potential $V_0(y)$. Find the potential inside the "slot".



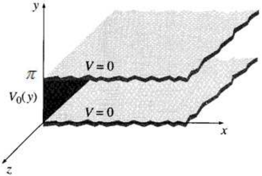
Boundary Conditions



- Solve Laplace's Equation for potential $V(x, y, z)$ in the "Slot" $\mathcal{D}$:

$$\mathcal{D} = \{(x, y, z) | x > 0,\ 0 < y < \pi, -\infty < z < \infty\}$$

Boundary Conditions

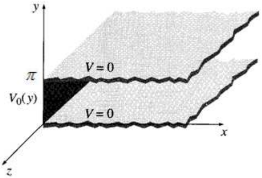


- Solve Laplace's Equation for potential $V(x, y, z)$ in the "Slot" $\mathcal{D}$:

$$\mathcal{D} = \{(x, y, z) | x > 0,\ 0 < y < \pi, -\infty < z < \infty\}$$

- Region $\mathcal{D}$ enclosed by 6 Boundary surfaces:
  - $x = 0$ and $x = \infty$
  - $y = 0$ and $y = \pi$
  - $z = \pm\infty$

Boundary Conditions



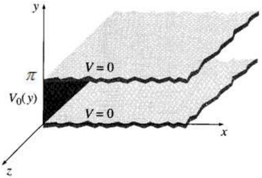
- Solve Laplace's Equation for potential $V(x, y, z)$ in the "Slot" $\mathcal{D}$:

$$\mathcal{D} = \{(x, y, z) | x > 0,\ 0 < y < \pi, -\infty < z < \infty\}$$

- Region $\mathcal{D}$ enclosed by 6 Boundary surfaces:
  - $x = 0$ and $x = \infty$
  - $y = 0$ and $y = \pi$
  - $z = \pm\infty$
- Translational symmetry in z: 2-dim Problem → V is independent of z:

$$V(x, y, z) \stackrel{2-\mathrm{dim}}{\rightarrow} V(x, y)$$

Boundary Conditions



- Solve Laplace's Equation for potential $V(x,y,z)$ in the "Slot" $\mathcal{D}$:

$$\mathcal{D} = \{(x,y,z)|x>0,\ 0<y<\pi, -\infty<z<\infty\}$$

- Region $\mathcal{D}$ enclosed by 6 Boundary surfaces:
  - $x=0$ and $x=\infty$
  - $y=0$ and $y=\pi$
  - $z=\pm\infty$
- Translational symmetry in z: 2-dim Problem $\rightarrow$ V is independent of z:

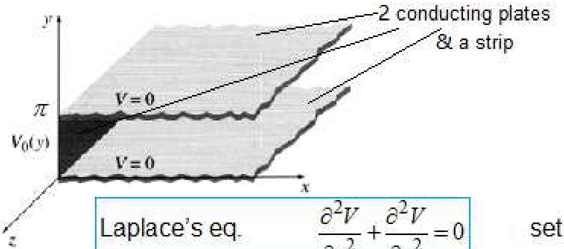
$$V(x,y,z) \stackrel{2-\mathrm{dim}}{\rightarrow} V(x,y)$$

- **4 Boundary Conditions**:

$$\begin{array}{rcl} (i) & V(x,y=0,z)=0 & \forall x,z \\ (ii) & V(x,y=\pi,z)=0 & \forall x,z \\ (iii) & V(x=0,y,z)=V_0(y) & \forall z \\ (iv) & V(x\rightarrow\infty,y,z)=0 & \forall y,z \end{array}$$

- No Boundary Conditions needed for the surfaces at $z=\pm\infty$.

Separation of Variables



(i) $V(y=0)=0$

(ii) $V(y=\pi)=0$

(iii) $V(x=0)=V_0(y)$

(iv) $V(x\to\infty)\to 0$

Laplace's eq. $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}=0$ set $\boxed{V(x,y)=X(x)Y(y)}$

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{x\ \text{depedent only}} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial^2 y}}_{y\ \text{depedent only}} = 0$$

$$\frac{d^2X}{dx^2}=k^2X \quad , \quad \frac{d^2Y}{dy^2}=-k^2Y$$

$$X(x)=Ae^{kx}+Be^{-kx} \quad , \quad Y(y)=C\sin ky+D\cos ky$$

$$\boxed{V(x,y)=\left(Ae^{kx}+Be^{-kx}\right)\left(C\sin ky+D\cos ky\right)}$$

Applying Boundary Conditions

$\boxed{\text{B.C.}}$ (iv) $V(x \to \infty) \to 0 \quad \Rightarrow \quad A = 0$ (where we take $k > 0$)

$\Rightarrow \quad V(x,y) = e^{-kx}\,(C \sin ky + D \cos ky)$ (B is absorbed)

$\boxed{\text{B.C.}}$ (i) $V(y = 0) = 0 \quad \Rightarrow \quad D = 0$

$\Rightarrow \quad V(x,y) = Ce^{-kx} \sin ky$

$\boxed{\text{B.C.}}$ (ii) $V(y = \pi) = 0 \quad \Rightarrow \quad \sin k\pi = 0 \quad \Rightarrow \quad \boxed{k = 1, 2, 3 \cdots \in \mathbb{N}}$

$$\boxed{V(x,y) = \sum_{k=1}^{\infty} C_k e^{-kx} \sin ky}$$

Principle of superposition due to Linearity of Laplace's Equation

$\boxed{\text{B.C.}}$ (iii) $V(x = 0) = V_0(y) \Rightarrow$ A fourier series with $\gamma = 0$ and $l = \pi/2$

$$\boxed{V_0(y) = \sum_{k=1}^{\infty} C_k \sin ky} \qquad \boxed{C_k = \frac{2}{\pi}\int_0^{\pi} V_0(y) \sin ky \, dy}$$

Use of Fourier Trick to find C_k:

- We obtained the following Fourier Series:

$$V_0(y) = \sum_{k=0}^{\infty} C_k \sin ky.$$

Multiplying both sides by $\sin py$ and integrating between $0 \leq y \leq \pi$:

$$\begin{aligned}
\int_0^{\pi} V_0(y) \sin py \, dy &= \int_0^{\pi} \left[\sum_{k=0}^{\infty} C_k \sin ky \sin py \right] dy \\
&= \sum_{k=0}^{\infty} C_k \left[\int_0^{\pi} \sin ky \sin py \, dy \right] \\
&= \sum_{k=0}^{\infty} C_k \left[\frac{\pi}{2} \delta_{pk} \right] = \frac{\pi}{2} C_p \\
C_p &= \frac{2}{\pi} \int_0^{\pi} V_0(y) \sin py \, dy.
\end{aligned}$$

General Solution: $$V(x,y) = \sum_{k=1}^{\infty} \left[\frac{2}{\pi} \int_0^{\pi} V_0(y) \sin ky \, dy \right] e^{-kx} \sin ky$$

Final Solution for b.c. $V_0(y) = V_0 = const.$

Example

For $V_0(y) = V_0 = constant$

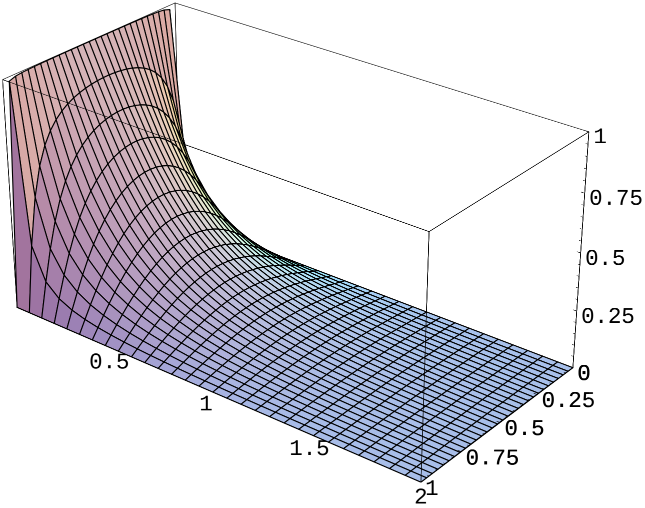
$$C_k = \frac{2V_0}{\pi}\int_0^\pi \sin ky \, dy$$

$$= \frac{2V_0}{k\pi}(1-\cos k\pi) = \begin{cases} 0 & if\ k = even \\ \frac{4V_0}{k\pi} & if\ k = odd \end{cases}$$

$$\boxed{V(x,y) = \frac{4V_0}{\pi}\sum_{k=1,3,5,\ldots}\frac{1}{k}e^{-kx}\sin ky = \frac{2V_0}{\pi}\tan^{-1}\left(\frac{\sin y}{\sinh x}\right)}$$

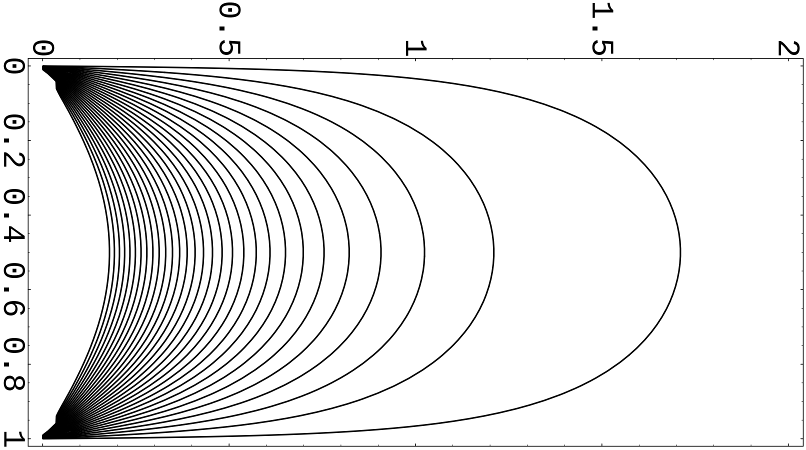
No matter what method (other than Separation of Variables) you use to solve this problem, you are guaranteed by **Uniqueness Theorem** to get the same answer!

Solution with b.c. $V_0(y) = V_0 = 1$



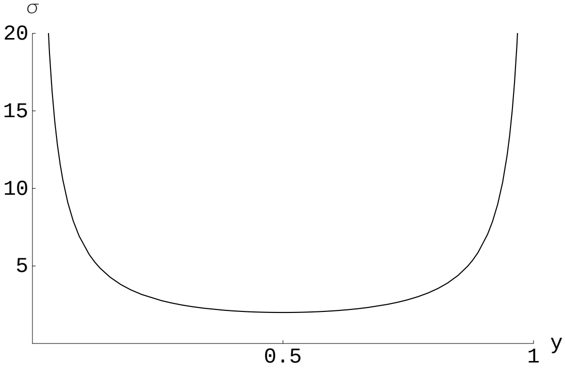
Electrostatic Potential $V\left(x, \frac{y}{\pi}\right)$ within the "slot"

Equipotentials with b.c. $V_0(y) = V_0 = 1$



Contour Plot of the Equipotentials of $V\left(x, \frac{y}{\pi}\right)$

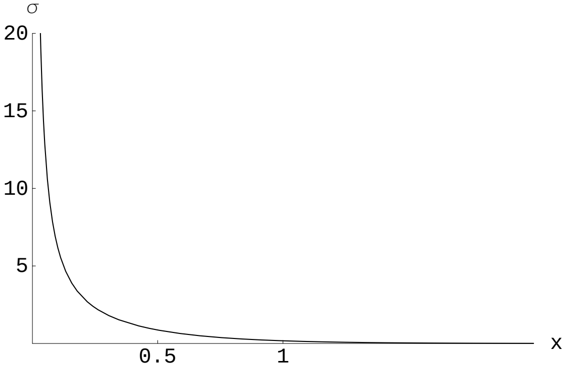
Surface charge density with $V_0(y) = V_0 = 1$



Induced charge density on the $x = 0$ plane or the end strip

$$\sigma\left(0, \frac{y}{\pi}\right) = \epsilon_0 \left(\mathbf{E} \cdot \hat{\mathbf{i}}|_{x=0}\right) = -\epsilon_0 \left.\frac{\partial V}{\partial x}\right|_{x=0}$$

Final Solution for $V_0(y) = V_0 = 1$

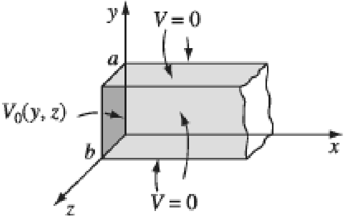


Induced charge density on the xz-plane at $y = 0$, i.e.,

$$\sigma(x, 0) = \epsilon_0 \left(\mathbf{E} \cdot \hat{\mathbf{j}}|_{y=0} \right) = -\epsilon_0 \left. \frac{\partial V}{\partial y} \right|_{y=0}$$

3D Laplace's Equation in Cartesian System

Example An infinitely long rectangular metal pipe (sides a and b) is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0(y, z)$. Find the potential inside the pipe.



$$\left.\begin{array}{ll}
\text{(i)} & V = 0 \text{ when } y = 0, \\
\text{(ii)} & V = 0 \text{ when } y = a, \\
\text{(iii)} & V = 0 \text{ when } z = 0, \\
\text{(iv)} & V = 0 \text{ when } z = b, \\
\text{(v)} & V \to 0 \text{ as } x \to \infty, \\
\text{(vi)} & V = V_0(y, z) \text{ when } x = 0.
\end{array}\right\} \textbf{BC}$$

This is a genuinely three-dimensional problem, $\boxed{\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0}$

$$V(x, y, z) = X(x)Y(y)Z(z) \quad \Longrightarrow \quad \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0.$$

Separation of Variables & Boundary Conditions

It follows that

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2, \quad \frac{1}{Z}\frac{d^2Z}{dz^2} = C_3, \quad \text{with } C_1 + C_2 + C_3 = 0.$$

Setting $C_2 = -k^2$ and $C_3 = -l^2$, we have $C_1 = k^2 + l^2$,

3 ODEs: $$\frac{d^2X}{dx^2} = (k^2 + l^2)X, \quad \frac{d^2Y}{dy^2} = -k^2Y, \quad \frac{d^2Z}{dz^2} = -l^2Z.$$

$$\Longrightarrow \quad \boxed{\begin{aligned} X(x) &= Ae^{\sqrt{k^2+l^2}\,x} + Be^{-\sqrt{k^2+l^2}\,x}, \\ Y(y) &= C\sin ky + D\cos ky, \\ Z(z) &= E\sin lz + F\cos lz. \end{aligned}}$$

Boundary condition (v) implies $A = 0$, (i) gives $D = 0$, and (iii) yields $F = 0$, whereas (ii) and (iv) require that $k = n\pi/a$ and $l = m\pi/b$, where n and m are positive integers. Combining the remaining constants, we are left with

$$\boxed{V(x, y, z) = Ce^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin(n\pi y/a)\sin(m\pi z/b).}$$

Use of Fourier Trick

$$\boxed{V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x} \sin(n\pi y/a)\ \sin(m\pi z/b)}$$

$\boxed{\text{B.C.}}$ (vi) :

$$V(0,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sin(n\pi y/a)\ \sin(m\pi z/b) = V_0(y,z)$$

Use Fourier Trick: multiply by $\sin(n'\pi y/a)\ \sin(m'\pi z/b)$,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \int_0^a \sin(n\pi y/a)\ \sin(n'\pi y/a)\,dy \int_0^b \sin(m\pi z/b)\ \sin(m'\pi z/b)\,dz$$

$$= \int_0^a \int_0^b V_0(y,z) \sin(n'\pi y/a)\ \sin(m'\pi z/b)\,dy\,dz.$$

$$\boxed{C_{n,m} = \frac{4}{ab}\int_0^a \int_0^b V_0(y,z)\sin(n\pi y/a)\ \sin(m\pi z/b)\,dy\,dz.}$$

Final Solution for b.c. $V_0(y,z) = V_0 = const.$

Example

For instance, if the end of the tube is a conductor at *constant* potential $V_0 = V_0(y,z)$

$$C_{n,m} = \frac{4V_0}{ab}\int_0^a \sin(n\pi y/a)\,dy \int_0^b \sin(m\pi z/b)\,dz$$

$$= \begin{cases} 0, & \text{if } n \text{ or } m \text{ is even,} \\ \dfrac{16V_0}{\pi^2 nm}, & \text{if } n \text{ and } m \text{ are odd.} \end{cases}$$

$$\boxed{V(x,y,z) = \frac{16V_0}{\pi^2} \sum_{n,m=1,3,5\ldots}^{\infty} \frac{1}{nm} e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x} \sin(n\pi y/a)\,\sin(m\pi z/b)}$$