Physics II

Electromagnetism (Lecture 14)

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Macroscopic Electric Fields In Dielectrics

- In vacuum, the TRUE Electric field $\mathbf{E}_{\text{True}} \equiv \mathbf{E}_{\text{vac}}$ is unambiguously calculated which in general has contributions both from distant free as well as bound charge distributions $\rho_{\text{tot}} = \rho_f + \rho_b$.

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- Then it becomes crucial to define a realistic MACROSCOPIC Field:

Definition

MACROSCOPIC Electric field: *It is defined as the space average field over an arbitrary macroscopic volume $\mathcal{V}$ of matter which is large enough to contain a statistically large number ($\gtrsim 10^4 - 10^5$) of atoms or molecules of that material, yet small enough compared to the dimensions of the material sample, in order to preserve all significant large-scale spatial variations in the field, i.e.,*

$$\boldsymbol{\mathcal{E}}(\mathbf{r}) \equiv \langle \mathbf{E}_{\rm True}(\mathbf{r}) \rangle_{\mathcal{V}} = \frac{1}{\mathcal{V}} \iiint_{\mathcal{V}} \mathbf{E}_{\rm True}(\mathbf{r}' - \mathbf{r}) dv',$$

where, for convenience, the integral is defined over a spherical region $\mathcal{V}$.

Macroscopic Electric Fields In Dielectrics (contd.)

The entire dielectric medium can be thought of being composed of sufficiently finely grained spherical Averaging volumes (like, close-packing of marbles), such that each spherical volume contains a statistically large number of atoms or molecules.

Macroscopic Fields and Potential In Dielectrics

- We henceforth work with Macroscopic Electric field $\boldsymbol{\mathcal{E}}(\mathbf{r}) \equiv \langle \mathbf{E}_{\rm True}(\mathbf{r})\rangle_{\mathcal{V}}$, in dielectrics, which is a conservative field derivable from a coresponding Macroscopic Potential $\mathscr{V}(\mathbf{r}) = \langle V_{\rm True}(\mathbf{r})\rangle_{\mathcal{V}}$, such that

$$\nabla \times \boldsymbol{\mathcal{E}}(\mathbf{r}) = \nabla \times [-\nabla \mathscr{V}(\mathbf{r})] = 0 \qquad \& \qquad \oint_{\rm Loop} \boldsymbol{\mathcal{E}}(\mathbf{r}) \cdot d\mathbf{r} = 0.$$

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- **Modified Gauss's Differential Law:**

$$\epsilon_0 \nabla \cdot \boldsymbol{\mathcal{E}}(\mathbf{r}) \quad = \quad \rho_{\mathrm{tot}}(\mathbf{r}) = \rho_b(\mathbf{r}) + \rho_f(\mathbf{r})$$

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- The field $\mathbf{D} \equiv \epsilon_0 \boldsymbol{\mathcal{E}} + \mathbf{P}$ is termed as the ELECTRIC DISPLACEMENT.

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- **Modified Gauss's Integral Law:**

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = \iiint_{\mathcal{V}} \nabla \cdot \mathbf{D} \, dv = \iiint_{\mathcal{V}} \rho_f \, dv = Q_{f,\,\rm encl}$$

where S is an arbitrary closed surface bounding a region of dielectric $\mathcal{V}$ with total enclosed free charge $Q_{f,\,\rm encl}$.

Modified Gauss's Law in Dielectrics: Summary

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \cdot \boldsymbol{\mathcal{E}} = \frac{\rho_{tot}}{\epsilon_0} = \frac{\rho_b + \rho_f}{\epsilon_0}$$

$$\oiint_{\text{surface}} \mathbf{D} \cdot d\mathbf{S} = Q_{f,\,\text{encl}}$$

$$\oiint_{\text{surface}} \boldsymbol{\mathcal{E}} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} Q_{tot,\,encl} = \frac{1}{\epsilon_0} (Q_b + Q_f)_{\text{encl}}$$

The Constitutive Relation: $\mathbf{D} = \epsilon_0 \boldsymbol{\mathcal{E}} + \mathbf{P}$

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Warning!

- Henceforth, we revert back to using the old symbol $\mathbf{E} \longleftrightarrow \boldsymbol{\mathcal{E}}$ for the Macroscopic Electric Field keeping in mind that it is NOT the same as the True Electric Field $\mathbf{E}_{\text{True}}$ within a dielectric which in general includes the Microscopic Background Field $\mathbf{E}_{\text{Micr}}$.
- For free space (vacuum), they are equivalent, i.e., $\boldsymbol{\mathcal{E}} \equiv \mathbf{E}_{\text{True}} \Rightarrow \mathbf{E}$.

Polarized Sphere

Example

Consider an uncharged dielectric sphere with a "frozen-in" Polarization $\mathbf{P} = \frac{k}{r}\hat{\mathbf{r}}$, where k is a constant. Find the Electric field as a function of r.

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- **Method I:** The bound volume & surface charge densities:

$$\rho_b(r) = -\nabla \cdot \left(\frac{k}{r}\hat{\mathbf{r}}\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2} \quad ; \quad \sigma_b = (\mathbf{P}\cdot\hat{\mathbf{r}})_{r=R} = \frac{k}{R}$$

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- Total bound charge:

$$Q_b = Q_b^{(\text{volume})} + Q_b^{(\text{surface})} = \iiint\limits_{\mathcal{V}} \rho_b(r) dv' + \oiint_S \sigma_b da'$$

$$= \int_0^R \left(-\frac{k}{r'^2}\right) 4\pi r'^2 dr' + \oiint \frac{k}{R} da' = -4\pi kR + 4\pi kR = 0$$

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- **Field outside sphere** $(r \geq R)$: Since $(Q_f + Q_b)_{encl} = 0$, then applying Gauss's Integral Law for $\mathbf{E}$:

$$\begin{aligned} \oint\!\!\oint_{\mathbb{S}(r\geq R)} \mathbf{E}\cdot d\mathbf{a}' &= \frac{1}{\epsilon_0} Q_{tot,\,encl}(r) = 0 \\ \mathbf{E}_{r\geq R}(\mathbf{r}) &= 0 \end{aligned}$$

Polarized Sphere (contd.)

- **Field inside sphere** ($r < R$): σ_b does not contribute in the bulk, then using Gauss's Integral Law for $\mathbf{E}$:

$$\oiint_{s(r<R)} \mathbf{E} \cdot d\mathbf{a}' = \frac{1}{\epsilon_0} Q_{tot,\ encl}(r) = \frac{1}{\epsilon_0} \iiint_{v} \rho_b(r) dv' = \frac{1}{\epsilon_0} \iiint_{v} \left(-\frac{k}{r'^2} \right) dv'$$

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$$E(4\pi r^2) = \frac{1}{\epsilon_0} \int_0^r \left(-\frac{k}{r'^2}\right) 4\pi r'^2 dr' \implies \mathbf{E}(\mathbf{r}) = -\frac{k}{\epsilon_0 r} \hat{\mathbf{r}}$$

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- **Method II**: Since $Q_{f,\, encl} = 0$, then applying Modified Gauss's Law for **D**:

$$\oiint_{\mathbb{S}} \mathbf{D} \cdot da' = Q_{f,\ encl} = 0 \implies \mathbf{D} = 0\,, \quad \forall r \text{ (everywhere)}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = 0 \implies \mathbf{E}(\mathbf{r}) = -\frac{\mathbf{P}}{\epsilon_0} = \begin{cases} 0 & \text{if} \quad r > R \\ -\frac{k}{\epsilon_0 r}\hat{\mathbf{r}} & \text{if} \quad r \leq R \end{cases} .$$

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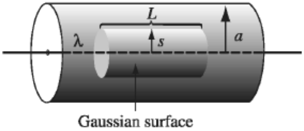
- Notice: Method II is much quicker without reference to bound charges!

Long Cylindrical Wire

Example

A long straight wire, carrying uniform line charge density λ, is surrounded by rubber insulation out to radius a. Find the Electric Displacements and Electric fields everywhere.



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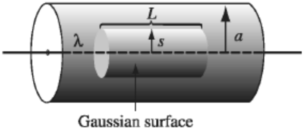

- Construct a coaxial cylindrical Gaussian surface S of radius s and length L:

$$\begin{aligned}
\oiint_S \mathbf{D} \cdot d\mathbf{S} &= Q_{f;\,\mathrm{encl}} = \lambda L \\
D(2\pi s L) &= \lambda L \\
\mathbf{D}(\mathbf{s}) &= \left(\frac{\lambda}{2\pi s}\right)\hat{\mathbf{s}}\,, \quad \forall s \text{ (everywhere)}.
\end{aligned}$$

- Note: This formula is applicable both inside and outside the cladding.

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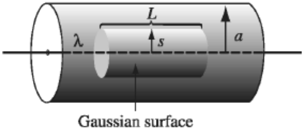

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- **Electric Field inside the cladding** ($s \leq a$): Polarization **P** as well as the dielectric constant being unknown, **E** can not be calculated.

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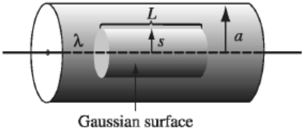

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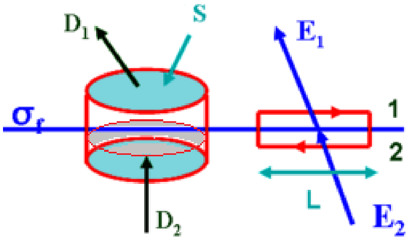
- **Electric Field inside the cladding** ($s \leq a$): Polarization $\mathbf{P}$ as well as the dielectric constant being unknown, $\mathbf{E}$ can not be calculated.
- **Electric Field outside the cladding** ($s > a$): Since Polarization $\mathbf{P} = 0$, so

$$\mathbf{D} \equiv \epsilon_0 \boldsymbol{E} + \cancelto{0}{\boldsymbol{P}} \quad \Longrightarrow \quad \mathbf{E}(\mathbf{s}) = \frac{1}{\epsilon_0}\mathbf{D}(\mathbf{s}) = \left(\frac{\lambda}{2\pi\epsilon_0 s}\right)\hat{\mathbf{s}}.$$

Boundary Conditions in Dielectrics



- Consider an interface of two dielectrics media (1 & 2) with total surface charge density $\sigma_{tot} = \sigma_f + \sigma_b$ at the interface and total volume charge densities $\rho_{tot,1} = \rho_{f1} + \rho_{b1}$ and $\rho_{tot,2} = \rho_{f2} + \rho_{b2}$, in the respectively bulks.
- Consider a pillbox-shaped Gaussian surface enclosing area S at the interface, with negligibly small width, $\epsilon \to 0$ in comparison with the base diameters.

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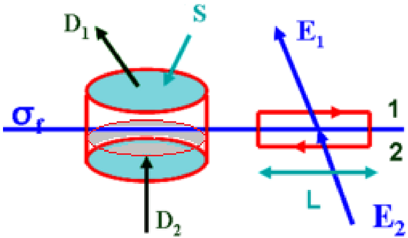

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- Consider a pillbox-shaped Gaussian surface enclosing area S at the interface, with negligibly small width, $\epsilon \to 0$ in comparison with the base diameters.
- Total enclosed **free** and **bound** charges within the Gaussian surface:

$$\begin{aligned} Q_{f,\,encl} &= \sigma_f S + \frac{\epsilon}{2}(\rho_{f1} + \rho_{f2})S \xrightarrow{\epsilon \to 0} \sigma_f S, \\ Q_{b,\,encl} &= \sigma_b S + \frac{\epsilon}{2}(\rho_{b1} + \rho_{b2})S \xrightarrow{\epsilon \to 0} \sigma_b S. \end{aligned}$$

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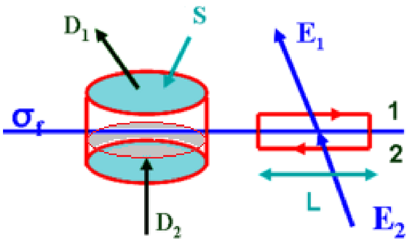

- Applying Gauss's Law for **D**:

$$\lim_{\epsilon\to 0} \oiint_S \mathbf{D}\cdot d\mathbf{S} = (\mathbf{D}_1\cdot\hat{\mathbf{n}}_1 + \mathbf{D}_2\cdot\hat{\mathbf{n}}_2)\, S \quad = \quad \lim_{\epsilon\to 0} Q_{f,\,encl} = \sigma_f\, S$$

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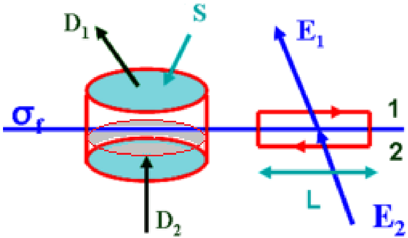

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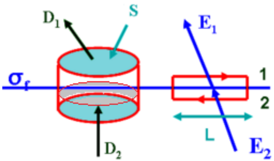
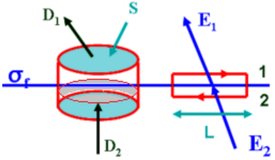


- Macroscopic Electric field being conservative in nature, the circulation of **E** around any closed loop must vanish. Choosing a narrow rectangular loop of length L and vanishing end widths $\epsilon \to 0$ straddling across the interface,

$$\lim_{\epsilon\to 0} \oint_{\text{Loop}} \mathbf{E}\cdot d\mathbf{l} = \mathbf{E}_1\cdot\mathbf{L} + \mathbf{E}_2\cdot(-\mathbf{L}) = 0$$

$$(\mathbf{E}_1 - \mathbf{E}_2)\cdot\hat{\mathbf{n}}_{1\,||} = 0$$

$$E_{1||} = E_{2||}$$

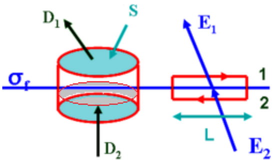


- Macroscopic Electric field being conservative in nature, the circulation of **E** around any closed loop must vanish. Choosing a narrow rectangular loop of length L and vanishing end widths $\epsilon \to 0$ straddling across the interface,

$$\begin{aligned} \lim_{\epsilon\to 0} \oint_{\text{Loop}} \mathbf{E}\cdot d\mathbf{l} = \mathbf{E}_1\cdot\mathbf{L} + \mathbf{E}_2\cdot(-\mathbf{L}) &= 0 \\ (\mathbf{E}_1 - \mathbf{E}_2)\cdot\hat{\mathbf{n}}_{1\,||} &= 0 \\ E_{1||} &= E_{2||} \end{aligned}$$

- Since $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$, it follows that $D_{||} = \epsilon_0 E_{||} + P_{||}$ and consequently

$$D_{1||} - D_{2||} = P_{1||} - P_{2||}.$$



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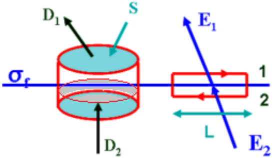
$$D_{1||} - D_{2||} = P_{1||} - P_{2||}.$$

- Similarly, from $D_\perp = \epsilon_0 E_\perp + P_\perp$

$$D_{1\perp} - D_{2\perp} = \epsilon_0(E_{1\perp} - E_{2\perp}) + (P_{1\perp} - P_{2\perp})$$



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(\mathbf{E}_1 - \mathbf{E}_2)\cdot\hat{\mathbf{n}}_{1\,||} &= 0 \\
E_{1||} &= E_{2||}
\end{aligned}$$

- Since $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$, it follows that $D_{||} = \epsilon_0 E_{||} + P_{||}$ and consequently

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- Similarly, from $D_\perp = \epsilon_0 E_\perp + P_\perp$

$$\begin{aligned}
D_{1\perp} - D_{2\perp} &= \epsilon_0(E_{1\perp} - E_{2\perp}) + (P_{1\perp} - P_{2\perp}) \\
\sigma_f &= (\sigma_f + \sigma_b) + (P_{1\perp} - P_{2\perp}) \\
P_{1\perp} - P_{2\perp} &= -\sigma_b.
\end{aligned}$$

Boundary Conditions in Dielectrics: SUMMARY

$$
\begin{aligned}
D_{1\perp} - D_{2\perp} &= \sigma_f, \\
D_{1||} - D_{2||} &= P_{1||} - P_{2||}, \\
P_{1\perp} - P_{2\perp} &= -\sigma_b, \\
E_{1\perp} - E_{2\perp} &= \frac{1}{\epsilon_0}(\sigma_f + \sigma_b), \\
E_{1||} &= E_{2||}, \\
V_1 &= V_2, \\
\frac{\partial V_1}{\partial n} - \frac{\partial V_2}{\partial n} &= \frac{1}{\epsilon_0}(\sigma_f + \sigma_b).
\end{aligned}
$$