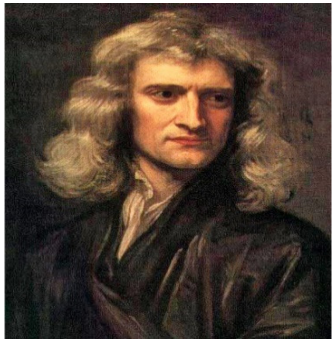
PH101

Lecture 10

Variational Calculus

History of Variational Calculus (Wikipedia/Rana&Joag)

Pierre de Fermat (1607 –1665)



Isaac Newton (1642 – 1727)

Jacob Bernoulli (1655 – 1705)
Algebra

Johann (**Jean** or **John**) **Bernoulli** (1667 –1748)
Variational calculus

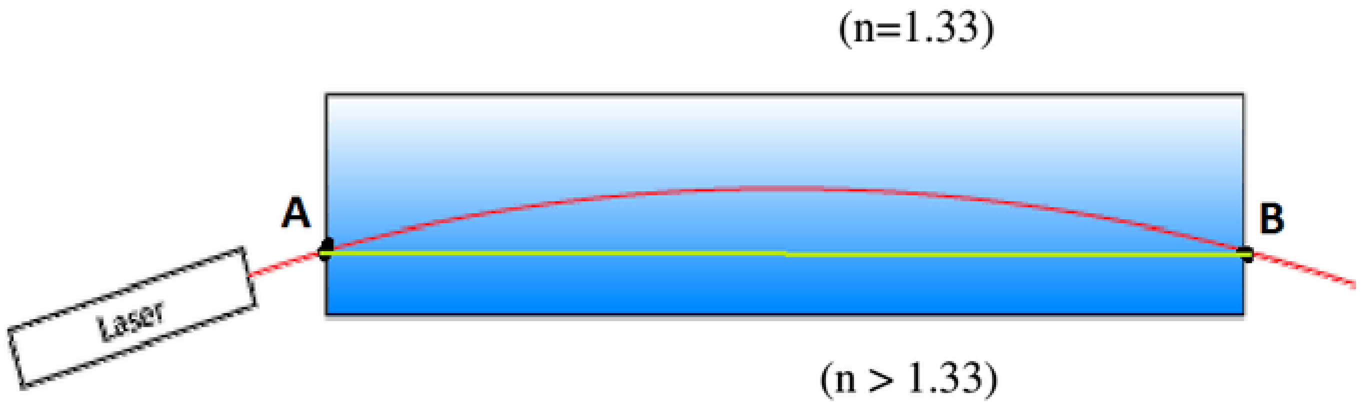
Leonhard Euler (1707-1783)

Joseph-Louis Lagrange

Daniel Bernoulli (1700 –1782)
Bernoulli's principle on fluids

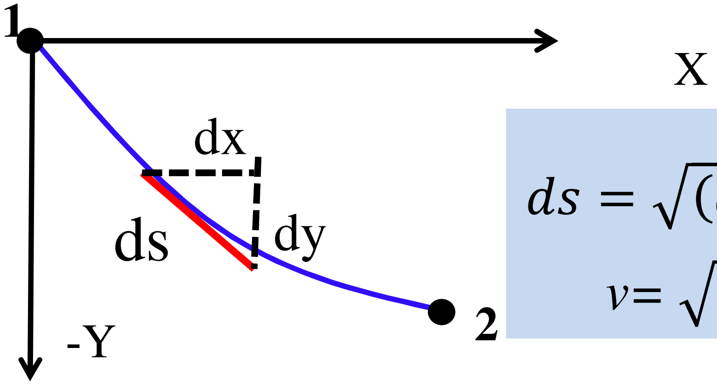
Fermat's principle of least (extremum) time ~1662

$$n = \frac{\text{speed of light in vacuum}}{\text{speed of light in medium}}$$



Jean Bernoulli’s challenge!
“Brachistochrone”

❑ What should be the shape of a stone’s trajectory (or, of a roller coaster track) so that released from point 1 it reaches point 2 in the shortest possible time? **Brachistochrone problem!** ~1696



Brachisto~ shortest **Chrone** ~ time

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{(1 + y'^2)}\,dx$$

$$v = \sqrt{2gy}$$

❑ Time (from 1 to 2) $I = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{\sqrt{(1+y'^2)}}{\sqrt{2gy}}\,dx = \int_1^2 \boldsymbol{F(y, y', x)dx}$

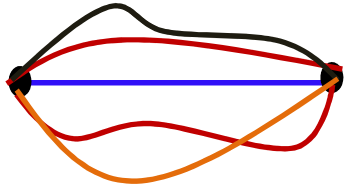
Wiki

Cycloid

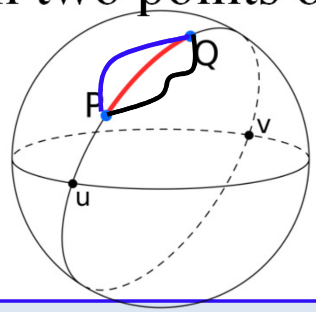
Extremums!

Can we prove mathematically

- ❑ Shortest distance between two points is a straight line?



- ❑ Shortest path between two points on the surface of a sphere is along the **great-circle**?



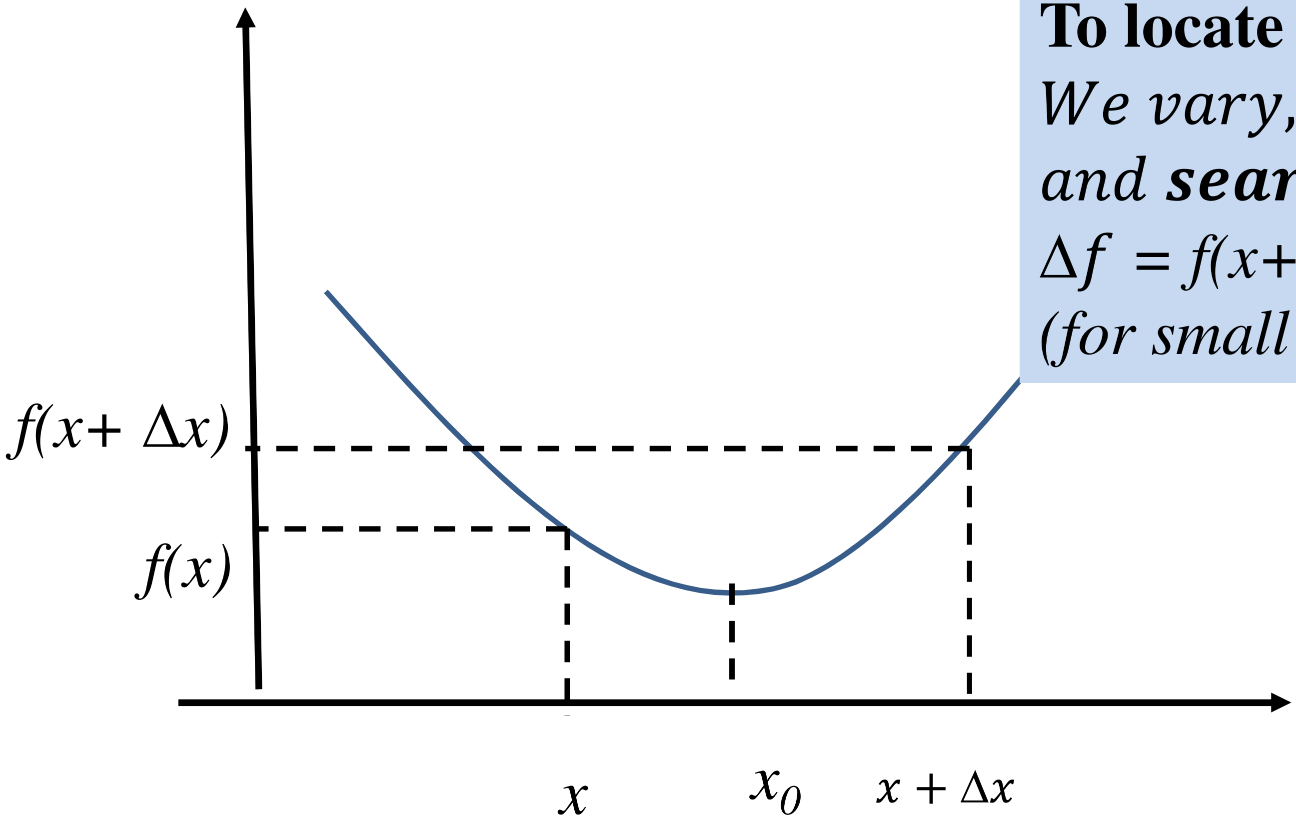
❑ To answer these questions, one need to know necessary condition that the integral $\mathbf{I} = \int_{x_1}^{x_2} F(y, y', x)dx$, where $y = y(x), y' = \frac{dy}{dx}$

is **stationary**

(**ie,** an **extremum! –** either a maximum or a minimum**!**).

❑ Interestingly we are already familiar with the solution!

Extremum of a function



To locate the minimum, $\boldsymbol{x_0}$

We vary, $\boldsymbol{x} \rightarrow$

and ***search*** *for the location*

$\Delta f = f(x+\Delta x) - f(x) \rightarrow 0$

(for small Δx*)*

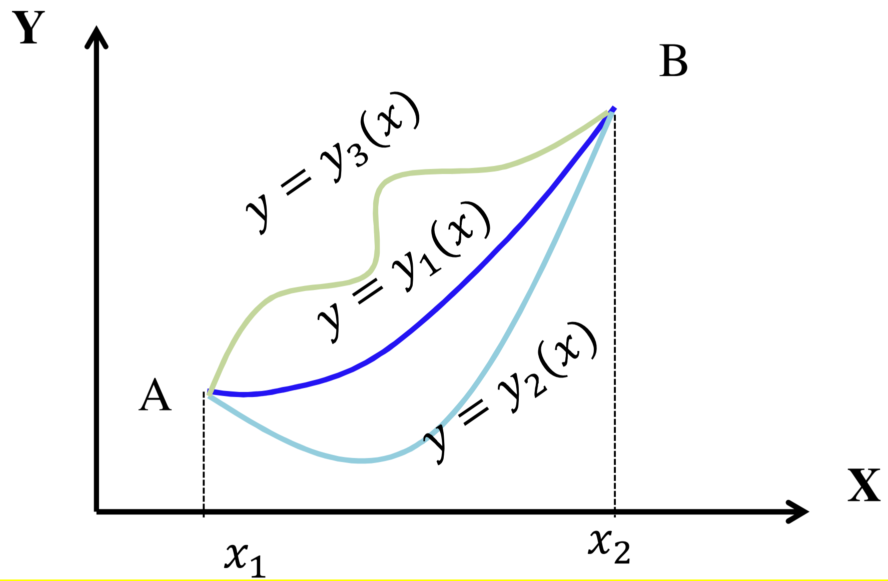
We say the function is **stationary** at, $\boldsymbol{x_0}$

(Meaning, for small steps, Δx, *at* $\boldsymbol{x_0}$ *the value of the function does not change.* $\Delta f = \left(\frac{\partial f}{\partial x}\right)_{x0} \Delta x = 0$

Possible integration paths

- Out of the **infinite** number of **possible paths,** $y(x)$,
  which path makes the integral,

$$\mathbf{I} = \int_{x_1}^{x_2} F(y, y', x)\,dx \quad \text{-}\textit{stationary}?$$



We need to find the condition for an **integral to be stationary,** where the **variable is a "function" itself [$y = y(x)$] -the integration path!**

Smart choice of varied paths

- **Step 1:** Let's assume $y(x) = Y(x)$ as the path for which integral,

$$\mathrm{I} = \int_{x_1}^{x_2} F(y, \dot{y}, x)dx \text{ is } \textbf{stationary}.$$

- **Step2:** $y(x) = Y(x) + \Delta y(x)$ can represent all possible paths between x_1 and x_2 for different $\Delta y(x)$.

Can you choose suitable mathematical form of $\Delta y(x)$ such that

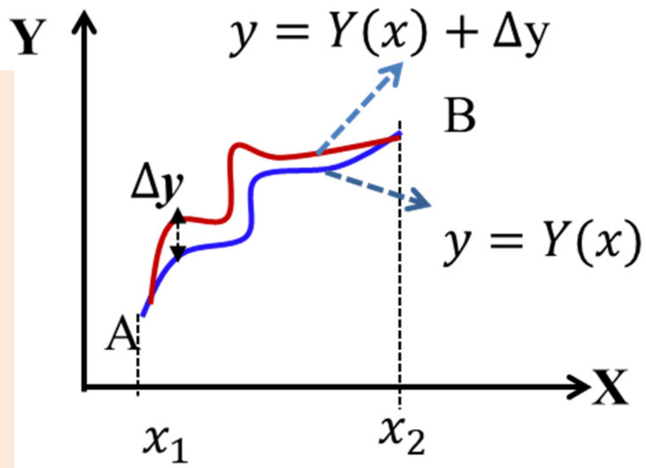
(i) $y(x) = Y(x) + \Delta y(x)$ should represent all varied paths but must not have variations at $\mathrm{A}(x_1)$ and $B(x_2)$ (fixed points).

(ii) $\Delta y(x)$ goes to zero in the limiting case when the varied paths are very close to $Y(x)$.

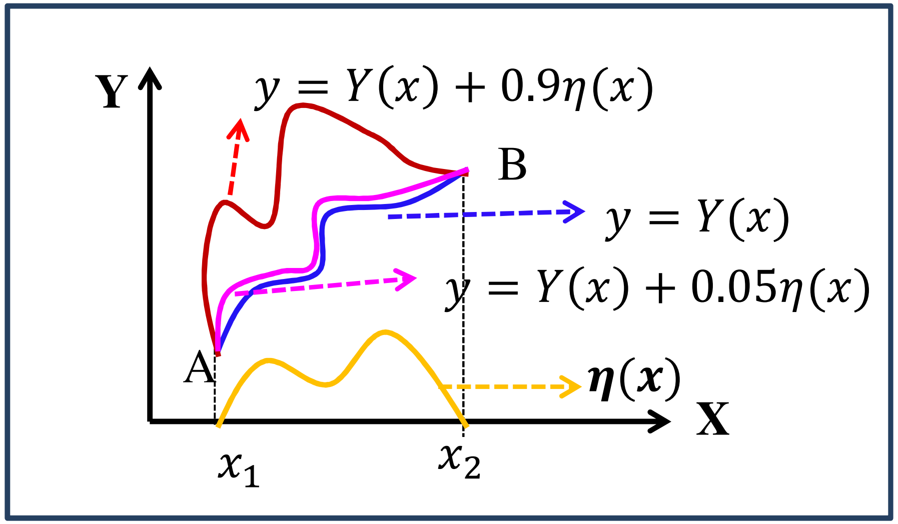
Let's check this choice $\Delta y(x) = \epsilon\, \eta(x)$

- where $\eta(x)$ is any **arbitrary function** of x such that $\eta(x_1) = \eta(x_2) = 0$. [condition (i) satisfied]

- ϵ is a parameter which can vary from 0 to higher value continuously. If we take **limit $\epsilon \to 0$**, then condition (ii) satisfied.

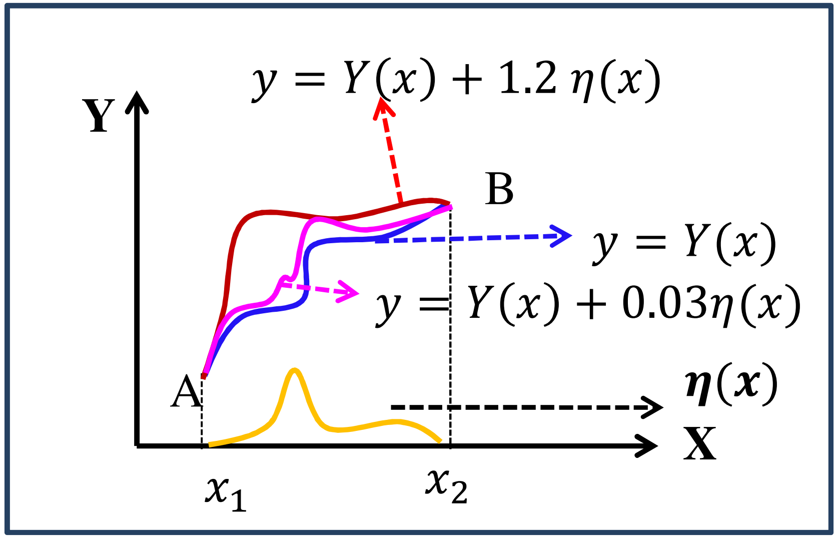


$\eta(x)$ and ϵ are indeed smart choice



Typical choice of arbitrary function $\eta(x)$,

- $\eta(x_1) = \eta(x_2) = 0.$
- By varying ϵ, different strength of $\eta(x)$ can be added to $Y(x)$ to generate range of possible paths between A and B.
- $\epsilon \to 0$ gives us the true path $Y(x)$.



For another choice of $\eta(x)$ to generate another series of possible paths between A and B by varying ϵ.

Thus arbitrary $\eta(x)$ and ϵ can produce all possible paths.

Stationary condition of integral

Step 3: Variation of the integral value for different paths nearby to the stationary path $Y(x)$ [$ie,\ \epsilon \to 0$ $hence$ $\Delta y \to 0$], is negligibly small.

The meaning of the statement is

Integration $\mathbf{I} = \int_{x_1}^{x_2} \boldsymbol{F(Y, Y', x)dx}$ along stationary path y(x) $= Y(x)$

and

$\boldsymbol{integration}$ along the **nearby paths** [$\boldsymbol{y(x,\epsilon)} = Y + \Delta y = \boldsymbol{Y(x)} + \boldsymbol{\epsilon}\eta(\boldsymbol{x})$, $\boldsymbol{and\ \epsilon \to 0}$]

$$\mathbf{I}(\boldsymbol{\epsilon}) = \int_{x_1}^{x_2} \boldsymbol{F\{(Y + \Delta y), (Y' + \Delta y'), x\}dx}] = \int_{x_1}^{x_2} \boldsymbol{F\{y(x,\epsilon)}, y'(\boldsymbol{x,\epsilon}), \boldsymbol{x\}dx}$$

must be equal. i.e, $\delta I(\boldsymbol{\epsilon}) = 0, \epsilon \to 0$

This can be achieved by, $\left.\boldsymbol{\frac{dI(\epsilon)}{d\epsilon}}\right|_{\boldsymbol{\epsilon \to 0}} = \mathbf{0}$

For stationary path $\left.\frac{d\boldsymbol{I}(\boldsymbol{\epsilon})}{d\boldsymbol{\epsilon}}\right|_{\boldsymbol{\epsilon}\to\boldsymbol{0}} = \boldsymbol{0}$

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon\to 0} = \frac{d}{d\epsilon}\left[\int_{x_1}^{x_2} F\{y(x,\epsilon), y'(x,\epsilon), x\}dx\right]$$

$$= \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\frac{\partial y}{\partial \epsilon} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \epsilon}\right)dx = \int_{x_1}^{x_2}\frac{\partial F}{\partial y}\eta\, dx + \int_{x_1}^{x_2}\frac{\partial F}{\partial y'}\eta' dx$$

Where,
$\boldsymbol{y}(\boldsymbol{x},\boldsymbol{\epsilon}) = (\boldsymbol{Y} + \boldsymbol{\epsilon}\eta)$
$y'(\boldsymbol{x},\boldsymbol{\epsilon}) = \boldsymbol{Y}' + \boldsymbol{\epsilon}\eta'$

$$= \int_{x_1}^{x_2}\frac{\partial F}{\partial y}\eta\, dx + \left.\frac{\partial F}{\partial y'}\eta\right|_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta dx$$

Integration by parts

Using $\left.\frac{\boldsymbol{\partial F}}{\boldsymbol{\partial y'}}\eta\right|_{\boldsymbol{x_1}}^{\boldsymbol{x_2}}$ =0

As $\eta(x_1) = \eta(x_2) = 0$

$$= -\int_{x_1}^{x_2}\left[\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}\right]\eta\, dx = 0$$

This equation is true for any possible choice of $\eta(\boldsymbol{x})$, thus

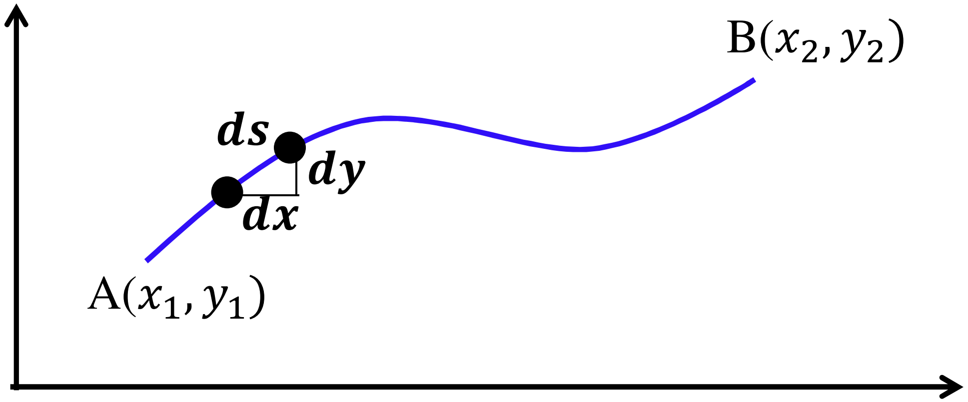
$\frac{\boldsymbol{d}}{\boldsymbol{dx}}\left(\frac{\boldsymbol{\partial F}}{\boldsymbol{\partial y'}}\right) - \frac{\boldsymbol{\partial F}}{\boldsymbol{\partial y}}$ =0,

Euler-Lagrange equation!

This is necessary condition for $\mathrm{I} = \int_{x_1}^{x_2} F(y, \dot{y}, x)dx$ to be stationary!

Application of Variational principle: Example1

❑ Given two points in a plane, what is the shortest path between them? We certainly know the answer: Straight line. Let's prove it using variation method



❑ Consider an arbitrary path $y(x)$, elementary length

❑ $ds = \sqrt{(dx)^2 + (dy)^2} = \left[\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\right]^{1/2} dx = \sqrt{(1 + y'^2)}\, dx$

❑ Total path length $\int_A^B ds = \int_{x_1}^{x_2} \sqrt{(1 + y'^2)}\, dx$

❑ Necessary condition for this integral to be stationary (maximum)

$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}$ =0; Here $F(y, y', x) = (1 + y'^2)^{1/2}$

Application of variational principle: Example1

$$\frac{\partial F}{\partial y'} = \frac{\partial}{\partial y'}\left\{\sqrt{(1+y'^2)}\right\} = \frac{y'}{\sqrt{(1+y'^2)}}$$

$$\frac{\partial F}{\partial y} = 0 \rightarrow \frac{y'}{\sqrt{(1+y'^2)}} = \mathrm{A} \quad \text{const.}$$

Thus

$$y'^2 = A^2(1+y'^2)$$

$$y'^2(1-A^2) = A^2$$

$$y' = \sqrt{\frac{A^2}{(1-A^2)}} = m$$

$$y(x) = mx + C,$$

$$\textit{Where } m \textit{ and } C \textit{ are constant}$$

Equation of straight line.

- Shortest distance between two points in a plane is straight line.

Questions?