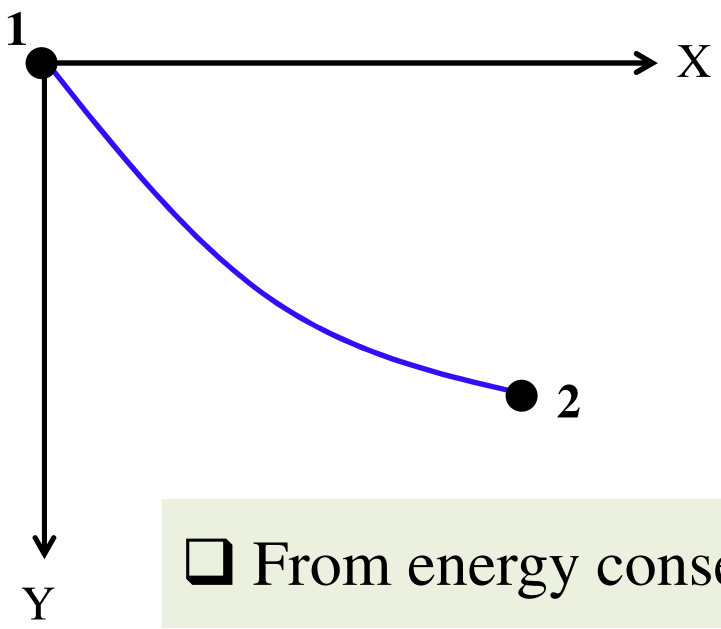
PH101

Lecture 11

Qiuz-I solution
Brachistochrone

Application of variation principle: Example 2

❑ What should be the tailored trajectory of a mass, m, which when released from point 1 will reach point 2 in the shortest possible time? **Brachistochrone problem**



❑ Time to travel from 1 to 2

$$\text{Time } (1 \rightarrow 2) = \int_1^2 \frac{ds}{v}$$

$$ds \rightarrow Elementary\ path\ length$$
$$v \rightarrow Instantaneous\ velocity$$

❑ From energy conservation, $\frac{1}{2}mv^2 = mgy\,;\, v = (2gy)^{1/2}$

$$ds = [(dx)^2 + (dy)^2]^{1/2} = \left[\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}\right]^{1/2} dy = \left(1 + x'^2\right)^{1/2} dy$$

$$\boldsymbol{x' = \frac{dx}{dy}}$$

Application of variation principle: Brachistochrone problem

- Time to travel from 1 to 2

Time1 → 2)

$$= \int_1^2 \frac{ds}{v} = \int_0^{y_2} \frac{\left(1 + \boldsymbol{x'}^2\right)^{1/2}}{(2gy)^{1/2}} dy = \int_0^{y_2} F\{x(y), \boldsymbol{x'}(y), y\} dy$$

What about $F\{y(x), y'(x), x\}$?

Then $\frac{\partial F}{\partial y} \neq 0$

Mathematically harder!

Where,

$$F\{x(y), \boldsymbol{x'}(y), y\} = \frac{\left(1 + \boldsymbol{x'}^2\right)^{1/2}}{(2gy)^{1/2}}$$

- Necessary condition for the integral (total time) to be extremum

$$\frac{\boldsymbol{d}}{\boldsymbol{dy}}\left(\frac{\boldsymbol{\partial F}}{\boldsymbol{\partial x'}}\right) - \frac{\boldsymbol{\partial F}}{\boldsymbol{\partial x}} = 0$$

Application of variation principle: Brachistochrone problem

$$\frac{\partial F}{\partial \boldsymbol{x}'} = \frac{\partial}{\partial \boldsymbol{x}'}\left\{\frac{\left(1+\boldsymbol{x}'^2\right)^{1/2}}{(2gy)^{1/2}}\right\} = \frac{\boldsymbol{x}'\left(1+\boldsymbol{x}'^2\right)^{-1/2}}{(2gy)^{1/2}}; \qquad \frac{\partial F}{\partial x} = 0$$

$$\frac{d}{dy}\left[\frac{\boldsymbol{x}'\left(1+\boldsymbol{x}'^2\right)^{-1/2}}{(2gy)^{1/2}}\right] = 0; \text{ Hence } \frac{\boldsymbol{x}'\left(1+\boldsymbol{x}'^2\right)^{-1/2}}{(2gy)^{1/2}} = Constant;$$

$$\frac{\boldsymbol{x}'^2}{y\left(1+\boldsymbol{x}'^2\right)} = Constant = \frac{1}{2a}$$

- To solve the integral, substitute $y = a(1-\cos\theta) \ldots\ldots (1)$

$$\text{thus } dy = a\sin\theta\, d\theta$$

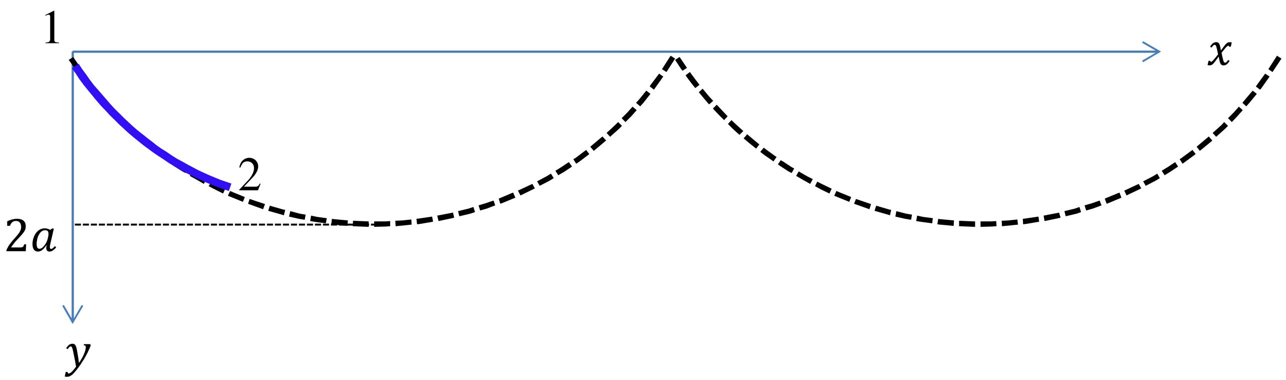
$$x = \int \sqrt{\frac{a(1-\cos\theta)}{a(1+\cos\theta)}}\, a\sin\theta\, d\theta = a\int\sqrt{\frac{(1-\cos\theta)}{(1+\cos\theta)}}\sqrt{(1-\cos\theta)(1+\cos\theta)}\, d\theta$$

$$x = a\int(1-\cos\theta)\, d\theta; \qquad x = a(\theta - \sin\theta) + constant \ldots (2)$$

Application of variation principle: Brachistochrone problem

❑ $y = a(1 - \cos\theta)$; $x = a(\theta - \sin\theta) + constant$
These two equations are parametric equation of the required path.

According to choice, initial point $x = y = 0$;
Thus $y = a(1 - \cos\theta)$; $x = a(\theta - \sin\theta)$ required path



❑ Path is a part of cycloid the curve traced out by a point on the rim of a wheel of radius a, rolling along the underside of the x axis.