PH101

Lecture 12

Principle of Least Action

- $L(q_j, \dot{q}_j, t)$ →Lagrangian of system of particles

- **Action integral** → $\int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)dt$

- A mechanical system will evolve in time in such that action integral is stationary → **Hamilton's Principle of Least Action**

$$\int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)dt \rightarrow \textbf{Stationary} \rightarrow \delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)dt = 0$$

- Stationary condition of Action integral

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

→ Lagrange's equation from Variational principle

symmetries and conservation laws

Cyclic coordinates, symmetries and conservation laws

Cyclic coordinate (q_j)

Corresponding generalized (canonical) momentum ($p_j = \frac{\partial L}{\partial \dot{q}_j}$) conserved

L is independent of the particular cyclic coordinate (q_j).
$q_j \rightarrow q_j + \delta q_j$ has no effect on L.

System is symmetric under **translation in generalized coordinate q_j**

- **Symmetry in generalized coordinate gives rise to a conserved canonical momentum.**

Example: For planetary motion

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}$$

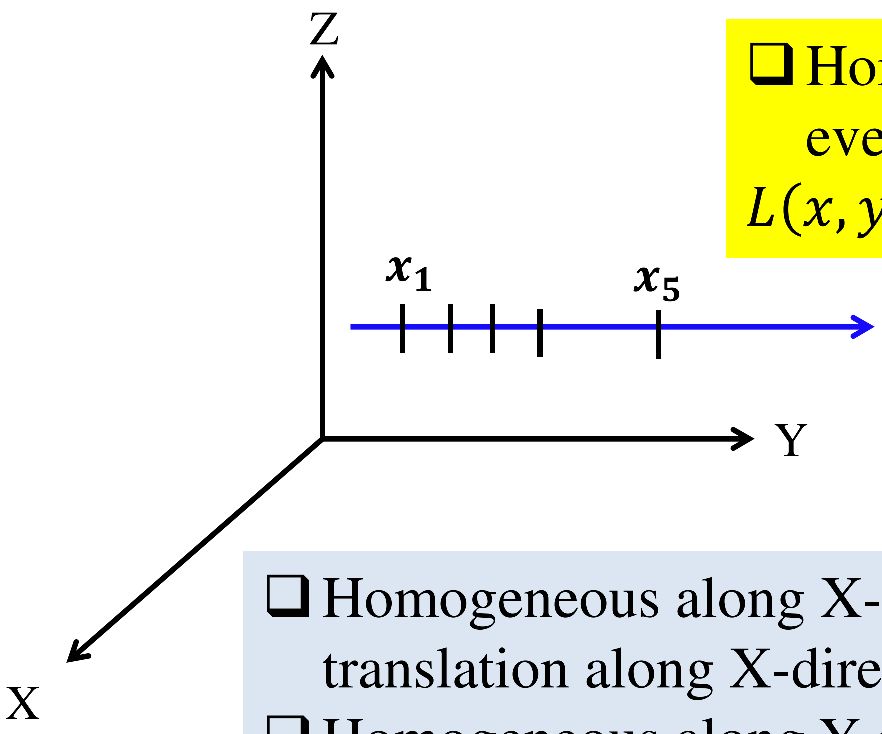
L is independent of rotation angle θ(cyclic coordinate), the system has rotational symmetry [symmetric under translation in $\theta \rightarrow \theta + \delta\theta$) , the system remains the same after change in θ.

As θ is cyclic, corresponding generalized (canonical) momentum $p_\theta = mr^2\dot{\theta}$=Constant; which is nothing but angular momentum.

Conclusion: Conservation of angular momentum is related to rotational symmetry of the system.

Translational symmetry and homogeneity of space

- **Homogeneity of space**: Space is such that in a particular direction, all the points are equivalent.



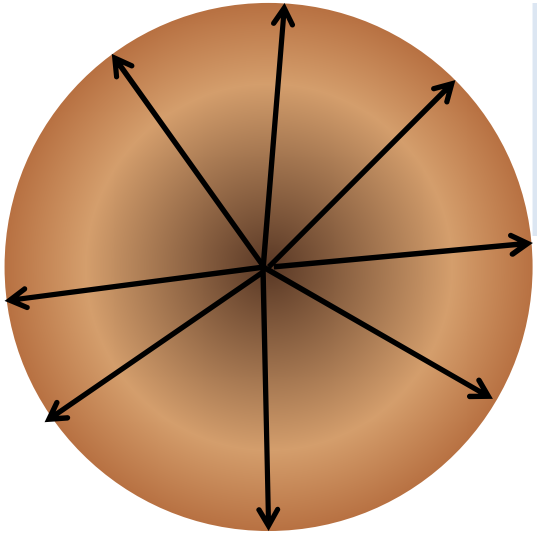
- Homogeneous in X-direction→Potential at every point along the line

$L(x, y, z) = L(x', y, z)$

- Homogeneous along X-direction→ L is invariant under translation along X-direction→ p_x conserved
- Homogeneous along Y-direction→ L is invariant under translation along Y-direction→ p_y conserved.
- Homogeneous along Z-direction→ L is invariant under translation along Z-direction→ p_z conserved

Rotational symmetry and isotropy of space

- **Isotropy of space**: Different directions around a point are all equivalent (at the same distance from that point).



- Thus→ $L(r, \theta, \varphi, \psi) = L(r, \theta', \varphi', \psi)$, function of r only.

Isotropy of space≡ Rotational symmetry of the system in both θ, φ and ψ

- All the directions are equivalent
- Potential energy in different directions (at the same distance from a particular point) must be same, as all directions are equivalent.
- Thus $L(r)$, *independent of* θ, φ and ψ. , L is invariant under rotation.
- p_φ and p_θ and p_ψ are conserved.

Symmetry and conservation laws

Homogeneity of space
(Translational symmetry) ⇒ **Conservation of linear momentum**

Isotropy of space
(Rotational symmetry) ⇒ **Conservation of angular momentum**

Homogeneity in time?

$$L(q_j, \dot{q}_j, t) = L(q_j, \dot{q}_j, t')$$

Only possible if L does not have explicit time dependence

Homogeneity in time
("translation" in time) ⇒ **Conservation of energy**

- If L does not explicitly depend on time, then energy of the system is conserved, provided potential energy is velocity independent.

Homogeneity in time leads to energy conservation: Proof

- $L = L(q_1, \dots q_n, \dot{q}_1, \dots . \dot{q}_n, t)$

- Using the chain rule of partial differentiation

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial t}$$

- Using Lagrange's eqn.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L\right) + \frac{\partial L}{\partial t} = 0 \quad \text{------------------------} \quad [1]$$

- If L does not have explicit time dependence

$$i,e \ \frac{\partial L}{\partial t} = 0$$

$$\sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L = Constant \quad \text{------------} \quad 2$$

$$\sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j = 2T$$

- If V does not depend on generalized velocity, $\frac{\partial V}{\partial \dot{q}_j} = 0; \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$ as $L = T - V$

- If L does not explicitly depend on time ($\frac{\partial L}{\partial t} = 0$) and V is velocity independent,

$$\sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j - L = Constant$$

- For a single free particle $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$\sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = m(\dot{x}\dot{x} + \dot{y}\dot{y} + \dot{z}\dot{z}) = 2T$$

- The relationship is true for General case as well,

Euler's theorem: If $f(x_i)$ is a homogeneous function of the n_{th} degree of set of variables x_i, then $\boldsymbol{\sum_j \frac{\partial f}{\partial \dot{x}_j}\dot{x}_j = nf.}$

- Kinetic energy T is a function of 2nd degree of generalized velocities $\dot{q}_j$

Proof continue…

$$\sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j - L = Constant$$

Now, $$\sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = 2T$$

$$(2T - L) = constant$$
$$(2T - T + V) = constant$$
$$T + V = constant$$

Conclusion, $T + V = E = constant$,
if L does not explicitely depend on time and potential is velocity indepdent

- ❑ Total energy conserved if

L does not depend not explicitly depend on time→Change in time does not cause any change in the form of L →Homogeneity in time

Summary

❑ **Principle of least action**: Action I= $\int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)dt$ is stationary

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

❑ If L does not have explicit time dependence, i,e $L = L(q_1, \ldots q_n, \dot{q}_1, \ldots . \dot{q}_n)$

$$\sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L = Constant$$

❑ I = $\int_{x_1}^{x_2} F(x, y, y')dx$ stationary=> $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}$ =0

If F does not have explicit dependence on x, i,e $F = F(y, y')$

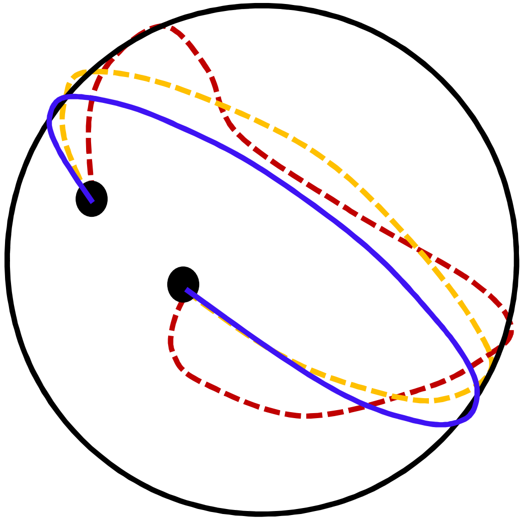
$$\frac{\partial F}{\partial y'}y' - F = Constant$$

Homogeneity in space — **Conservation of linear momentum**

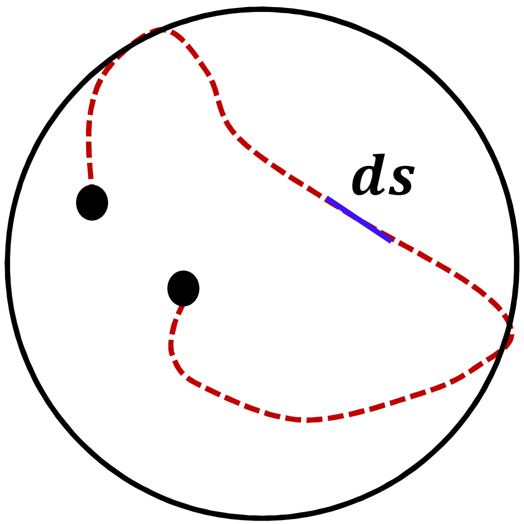
Isotropy of space — **Conservation of angular momentum**

Homogeneity in time — **Conservation of energy**

Application of variation principle: Shortest path between two points on the surface of a sphere



- Shortest path is the path along the great circle connecting the two points



- Elementary length (ds) between two points in spherical polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta \, d\varphi^2$$

- On the surface of the sphere,

$$r = R = constant$$

$$\dot{r} = 0$$

$$ds^2 = R^2 d\theta^2 + R^2 sin^2\theta \, d\varphi^2$$

Shortest path between two points on the surface of a sphere

- Total length between two points 1&2

$$S = \int_1^2 ds = \int_1^2 \sqrt{R^2 d\theta^2 + R^2 sin^2\theta\, d\varphi^2}$$

$$S = R\int_{\theta_1}^{\theta_2} \sqrt{1 + sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2}\, d\theta$$

Mathematically difficult due to non-zero $\frac{\partial F}{\partial \theta}$

- You can also express as

$$S = R\int_{\varphi_1}^{\varphi_2} \sqrt{sin^2\theta + \left(\frac{d\theta}{d\varphi}\right)^2}\, d\varphi$$

$$F\{\theta, \varphi(\theta), \varphi'(\theta)\} = \sqrt{1 + sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2} = \sqrt{1 + sin^2\theta \varphi'^2}$$

$$\varphi' = \frac{d\varphi}{d\theta}$$

- Necessary condition for the integral (total time) to be extremum

$$\frac{d}{d\theta}\left(\frac{\partial F}{\partial \varphi'}\right) - \frac{\partial F}{\partial \varphi} = 0$$

Shortest path between two points on the surface of a sphere

$$F = \sqrt{1 + \sin^2\theta\, \varphi'^2}$$

$$\frac{\partial F}{\partial \boldsymbol{\varphi}} = \mathbf{0}$$

$$\frac{\partial F}{\partial \boldsymbol{\varphi}'} = \frac{\sin^2\theta\, \boldsymbol{\varphi}'}{\sqrt{1 + \sin^2\theta\, \varphi'^2}}$$

$$\frac{d}{d\theta}\left(\frac{\sin^2\theta\, \boldsymbol{\varphi}'}{\sqrt{1 + \sin^2\theta\, \varphi'^2}}\right) = \mathbf{0}; \quad \frac{\sin^2\theta\, \boldsymbol{\varphi}'}{\sqrt{1 + \sin^2\theta\, \varphi'^2}} = \boldsymbol{constant} = \boldsymbol{k}$$

$$\sin^4\theta\, \varphi'^2 = k^2\left(1 + \sin^2\theta\, \varphi'^2\right); \quad \varphi' = \pm \frac{k \csc^2\theta}{\sqrt{1 - k^2 \csc^2\theta}}$$

$$\varphi' = \pm \frac{k \csc^2\theta}{\sqrt{1 - k^2(1 + \cot^2\theta)}} = \pm \frac{k \csc^2\theta}{\sqrt{1 - k^2 - k^2\cot^2\theta)}}$$

$$\varphi' = \frac{\pm k}{\sqrt{1 - k^2}} \frac{\csc^2\theta}{\sqrt{1 - \frac{k^2}{1 - k^2}\cot^2\theta}}$$

Shortest path between two points on the surface of a sphere

$$d\varphi = \alpha \frac{csc^2\theta\, d\theta}{\sqrt{1-\alpha^2 cot^2\theta}}$$

$$d\varphi = \frac{dq}{\sqrt{1-q^2}}\,;\ \int d\varphi = \int \frac{dq}{\sqrt{1-q^2}}$$

$$\varphi = \sin^{-1} q + \beta;\ q = \sin(\varphi - \beta)$$

$$\boldsymbol{\alpha\, cot\theta = \sin(\varphi - \beta)} \ldots \ldots . . [\mathbf{1}]$$

Let $\alpha = \frac{\pm k}{\sqrt{1-k^2}}$ and $q = \alpha \cot\theta$

$$dq = \alpha\, csc^2\theta\, d\theta$$

$\beta \rightarrow Integration\ constant$

To understand the meaning of $\boldsymbol{equation\ 1}$, multiply both sides by R

$$\alpha\, R\, cot\theta = R \sin(\varphi - \beta)$$

$$\alpha R\, cos\theta = R\, sin\theta\, sin\varphi\, cos\beta\ - R\, sin\theta\, cos\varphi\, sin\beta$$

$$\alpha z = cos\beta\, y - \sin\beta\ x$$

$$\boldsymbol{sin\,\beta\ x - cos\beta\, y - \alpha z = 0}$$

→ Equation of a plane passing through origin

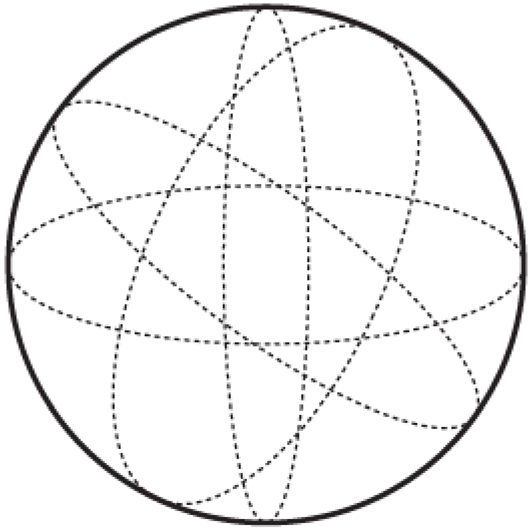
Equation of a plane passing through (x_0, y_0, z_0)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Shortest path between two points on the surface of a sphere

$$\sin\beta\ x - \cos\beta\ y - \alpha z = 0$$

This plane which passes through the origin slices through the sphere in great circles



Thus solution of Euler-Lagrange's equation are great circle routes

Shortest path between two points on the surface of a sphere must lie on this the great circle passing through those points.

Questions?