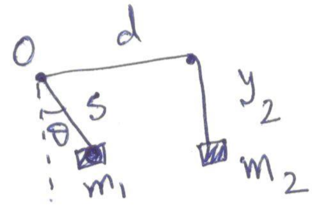
Q1. Report the degree of freedom (**DOF**) of the following systems in the space provided.
No justification **required!**

a)	Consider a ***thin*** uniform rigid rod of mass m and length l, under the following situations.		
[3×1]	The rod is sliding down such a way that one end of it always maintains contact with a vertical wall while the other end slips on a horizontal floor. **The whole motion is on the xy −plane.**	DOF=	1
	The rod is free to move any fashion but constrained ***only to the xy −plane.***	DOF=	3
	The rod is constrained to move ***inside*** a larger spherical shell of radius R ($2R > l$) such a way that both ends of the rod always maintain contact with the inner surface of the shell.	DOF=	3
b)	Water molecule (H_2O) is composed of two hydrogen atoms bonded to an oxygen atom, with its average O−H bonds measuring 1 Å, and with an equilibrium H−O−H angle of 104.5°. Obtain the DOF of a water molecule that is free to move in ***three dimensions*** under the following models.		
[4×1]	**Model#1:** The O−H bonds, or bond lengths, are flexible. But the H−O−H bond angle is fixed at 104.5°	DOF=	8
	Model#2: The O−H bonds are rigid at 1 Å, and the H−O−H bond angle is fixed at 104.5°.	DOF=	6
	Model#3: The O−H bonds are rigid at 1 Å, but the H−O−H bond angle is free to change.	DOF=	7
	Model#4: The O−H bonds as well as the H−O−H bond angle is free to change.	DOF=	9

Q2



$$y_2 = l - (s + d)$$

$$x_1 = s\sin\theta \;,\; y_1 = s\cos\theta$$

$$\dot{x}_1 = \dot{s}\sin\theta + s\cos\theta\,\dot{\theta} \;,\; \dot{y}_1 = \dot{s}\cos\theta - s\sin\theta\,\dot{\theta}$$

$$T = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m_2}{2}\dot{y}_2^2$$

$$= \frac{m_1}{2}\left(\dot{s}^2\sin^2\theta + s^2\cos^2\theta\,\dot{\theta}^2 + 2s\dot{s}\sin\theta\cos\theta\,\dot{\theta} + \dot{s}^2\cos^2\theta + s^2\sin^2\theta\,\dot{\theta}^2 - 2s\dot{s}\cos\theta\sin\theta\,\dot{\theta}\right) + \frac{m_2}{2}\dot{s}^2$$

$$\left[\text{as } \dot{y}_2 = -\dot{s}\right]$$

$$T = \frac{m_1}{2}\left(\dot{s}^2 + s^2\dot{\theta}^2\right) + \frac{m_2}{2}\dot{s}^2$$

$$V = -m_1 g y_1 - m_2 g y_2 = -m_1 g s\cos\theta - m_2 g(l-d-s)$$

$$L = T - V = \frac{m_1}{2}\left(\dot{s}^2 + s^2\dot{\theta}^2\right) + \frac{m_2}{2}\dot{s}^2 + s m_1 g\cos\theta + m_2 g(l-d-s)$$

$$\frac{\partial L}{\partial \dot{s}} = (m_1+m_2)\dot{s} \;;\; \frac{\partial L}{\partial s} = m_1 s\dot{\theta}^2 + m_1 g\cos\theta - m_2 g$$

EL① $\Rightarrow$ $(m_1+m_2)\ddot{s} - m_1 s\dot{\theta}^2 - m_1 g\cos\theta + m_2 g = 0$

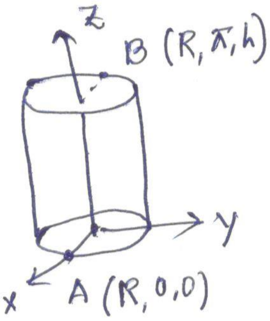
$$\frac{\partial L}{\partial \dot{\theta}} = m_1 s^2\dot{\theta} \;;\; \frac{\partial L}{\partial \theta} = -m_1 g s\sin\theta$$

EL② $\Rightarrow$ $m_1 s^2\ddot{\theta} + 2m_1\dot{\theta}s\dot{s} + m_1 g s\sin\theta = 0$

Q3

Elementary path length in cylindrical coordinate

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$



Thus elementary path length on the surface of a cylinder $ds^2 = R^2 d\theta^2 + dz^2$

Thus total path length from A to B [on the surface $r = R$, $dr = 0$]

$$= \int_A^B ds = \int_A^B \sqrt{R^2 d\theta^2 + dz^2} = \int_A^B \sqrt{R^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta$$

$$= \int_A^B \sqrt{R^2 + z'^2}\, d\theta = \int_A^B F(\theta, z, z')\, d\theta \quad \left\{ \text{where } z' = \frac{dz}{d\theta} \text{ and } F(\theta, z, z') = \sqrt{R^2 + z'^2} \right\}$$

path length to be extremum, condition is

$$\frac{d}{d\theta}\left[\frac{\partial F}{\partial z'}\right] - \frac{\partial F}{\partial z} = 0 \Rightarrow \frac{d}{d\theta}\left[\frac{\partial F}{\partial z'}\right] = 0 \quad \left[\text{as } \frac{\partial F}{\partial z} = 0\right]$$

$$\frac{\partial F}{\partial z'} = \text{constant} \Rightarrow \frac{z'}{\sqrt{R^2 + z'^2}} = A \text{ (constant)} \Rightarrow z'^2 = A^2 (R^2 + z'^2)$$

$$\Rightarrow \cancel{z'^2 (A} \Rightarrow z'^2 (1 - A^2) = A^2 R^2 \Rightarrow z' = \frac{AR}{\sqrt{1 - A^2}} \Rightarrow dz = \frac{AR}{\sqrt{1 - A^2}}\, d\theta$$

$$\Rightarrow z = \frac{AR}{\sqrt{1 - A^2}}\,\theta + C \text{ (constant)}$$

using b.c at A, $z = 0$, $\theta = 0$

$C = 0$, thus $z = \frac{AR}{\sqrt{1 - A^2}}\,\theta$.

using b.c at B, $z = h$, $\theta = \pi$

$$h = \frac{AR}{\sqrt{1 - A^2}}\,\pi$$

thus $\frac{AR}{\sqrt{1 - A^2}} = \frac{h}{\pi} \Rightarrow$ Hence $z = \frac{h}{\pi}\theta \Rightarrow z' = \frac{h}{\pi}$

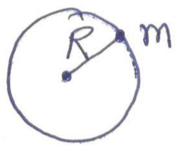
Hence total path length from A to B $= \int_A^B ds$

$$= \int_A^B \sqrt{R^2 + z'^2}\, d\theta = \int_0^\pi \sqrt{R^2 + \left(\frac{h}{\pi}\right)^2}\, d\theta$$

$$= \sqrt{R^2 + \left(\frac{h}{\pi}\right)^2}\,\pi = \sqrt{R^2 \pi^2 + h^2}$$

Q4

Kinetic energy in S.P coordinate



$$T = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

As the particle is constrained to move over the surface of sphere $r = R$; $\dot{r} = 0$

Hence $T = \tfrac{1}{2}m\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\phi}^2\right)$

As the particle is acted on by the forces of constrain, no gravitation force to be considered, $(V=0)$

Lagrangian $L = T - V = \tfrac{1}{2}m\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\phi}^2\right)$

Energy function $h = \sum_j p_j\dot{q}_j - L = P_\theta\dot{\theta} + P_\phi\dot{\phi} - L$

$$= P_\theta\dot{\theta} + P_\phi\dot{\phi} - \tfrac{1}{2}m\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\phi}^2\right) \quad \text{—(1)}$$

Now $P_\theta = \dfrac{\partial L}{\partial\dot{\theta}} = \dfrac{\partial}{\partial\dot{\theta}}\left[\tfrac{1}{2}m\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\phi}^2\right)\right] = mR^2\dot{\theta}$

Thus $\dot{\theta} = \dfrac{P_\theta}{mR^2}$

Similarly $P_\phi = \dfrac{\partial L}{\partial\dot{\phi}} = mR^2\sin^2\theta\,\dot{\phi} \Rightarrow \dot{\phi} = \dfrac{P_\phi}{mR^2\sin^2\theta}$

Hamiltonian of the system can be obtained from 'h'

$$H = P_\theta\frac{P_\theta}{mR^2} + P_\phi\frac{P_\phi}{mR^2\sin^2\theta} - \frac{1}{2}m\left\{R^2\left(\frac{P_\theta}{mR^2}\right)^2 + R^2\sin^2\theta\left(\frac{P_\phi}{mR^2\sin^2\theta}\right)^2\right\}$$

$$= \frac{P_\theta^2}{2mR^2} + \frac{P_\phi^2}{2mR^2\sin^2\theta}$$

Hamiltonian equations $\dot{p}_j = -\dfrac{\partial H}{\partial q_j}$; $\dot{q}_j = \dfrac{\partial H}{\partial p_j}$

$$\dot{P}_\theta = -\frac{\partial H}{\partial\theta} = \frac{P_\phi^2}{2mR^2}\times\frac{2\cos\theta}{\sin^3\theta} = \frac{P_\phi^2}{mR^2}\frac{\cos\theta}{\sin^3\theta}$$

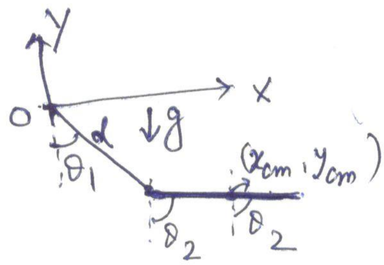
$$\dot{P}_\phi = -\frac{\partial H}{\partial\phi} = 0 \ ; \ P_\phi = \text{constant}$$

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{mR^2}$$

$$\dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{P_\phi}{mR^2\sin^2\theta}$$

Q5

Let coordinate of the C.M. $\Rightarrow (x_{cm}, y_{cm})$



$$\left.\begin{aligned} x_{cm} &= d\sin\theta_1 + \frac{l}{2}\sin\theta_2 \\ y_{cm} &= d\cos\theta_1 + \frac{l}{2}\cos\theta_2 \end{aligned}\right\} \begin{aligned} \dot{x}_{cm} &= d\cos\theta_1\,\dot\theta_1 + \tfrac{l}{2}\cos\theta_2\,\dot\theta_2 \\ \dot{y}_{cm} &= -d\sin\theta_1\,\dot\theta_1 - \tfrac{l}{2}\sin\theta_2\,\dot\theta_2 \end{aligned}$$

kinetic energy $T = \frac{1}{2}m\left[\dot{x}_{cm}^2 + \dot{y}_{cm}^2\right] + \frac{1}{2} I \dot\theta_2^2$

$$T = \tfrac{1}{2}m\left[\left(d\cos\theta_1\dot\theta_1 + \tfrac{l}{2}\cos\theta_2\dot\theta_2\right)^2 + \left(-d\sin\theta_1\dot\theta_1 - \tfrac{l}{2}\sin\theta_2\dot\theta_2\right)^2\right] + \frac{1}{2}\left(\frac{ml^2}{12}\right)\dot\theta_2^2$$

$$T = \tfrac{1}{2}m\left[d^2\dot\theta_1^2 + \frac{l^2}{4}\dot\theta_2^2 + l d\,\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)\right] + \frac{ml^2}{24}\dot\theta_2^2$$

$$T = \frac{1}{2}md^2\dot\theta_1^2 + \frac{1}{6}ml^2\dot\theta_2^2 + \tfrac{1}{2}mld\,\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)$$

potential energy $V = -mg\,y_{cm} = -mg\left(d\cos\theta_1 + \frac{l}{2}\cos\theta_2\right)$

Lagrangian $L = T - V$

$$= \tfrac{1}{2}md^2\dot\theta_1^2 + \frac{1}{6}ml^2\dot\theta_2^2 + \frac{1}{2}mld\,\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)$$
$$+ mg\left(d\cos\theta_1 + \tfrac{l}{2}\cos\theta_2\right)$$

Q6

Let coordinate of center of mass of the disc (x_{cm}, y_{cm}).
Coordinate of the particle (x, y).

Hence
$$\left.\begin{aligned} x &= x_{cm} + R\sin\theta \\ y &= y_{cm} - R\cos\theta = R - R\cos\theta \end{aligned}\right\} \quad \begin{aligned} \dot{x} &= \dot{x}_{cm} + R\cos\theta\,\dot{\theta} \\ \dot{y} &= +R\sin\theta\,\dot{\theta} \end{aligned}$$

Rolling without sliding condition

$$\dot{x}_{cm} = -R\dot{\theta}$$

Kinetic energy of the system
$$T = \tfrac{1}{2}(2m)\left[\dot{x}_{cm}^2 + \dot{y}_{cm}^2\right] + \tfrac{1}{2} I \dot{\theta}^2 + \tfrac{1}{2} m\left[\dot{x}^2 + \dot{y}^2\right]$$
$$T = m\left[(-R\dot{\theta})^2 + 0\right] + \tfrac{1}{2}\frac{(2m)R^2}{2}\dot{\theta}^2 + \tfrac{1}{2} m\left[(-R\dot{\theta} + R\cos\theta\,\dot{\theta})^2 + R^2\sin^2\theta\,\dot{\theta}^2\right]$$
$$T = mR^2\dot{\theta}^2 + \tfrac{1}{2}mR^2\dot{\theta}^2 + \tfrac{1}{2}mR^2\dot{\theta}^2 + \tfrac{1}{2}mR^2\dot{\theta}^2 - mR^2\dot{\theta}^2\cos\theta$$
$$= \tfrac{5}{2}mR^2\dot{\theta}^2 - mR^2\dot{\theta}^2\cos\theta.$$

potential energy $V = mgy + (2m)g\,y_{cm} = mgR(1-\cos\theta) + 2mgR.$

Lagrangian $L = T - V = \tfrac{5}{2}mR^2\dot{\theta}^2 - mR^2\dot{\theta}^2\cos\theta - mgR(1-\cos\theta) - 2mgR$

Lagrange's eqn: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) - \dfrac{\partial L}{\partial \theta} = 0$

$$\Rightarrow \frac{d}{dt}\left[5mR^2\dot{\theta} - 2mR^2\dot{\theta}\cos\theta\right] - \left[mR^2\dot{\theta}^2\sin\theta - mgR\sin\theta\right] = 0$$
$$\Rightarrow 5mR^2\ddot{\theta} - 2mR^2\ddot{\theta}\cos\theta + 2mR^2\dot{\theta}^2\sin\theta - mR^2\dot{\theta}^2\sin\theta + mgR\sin\theta = 0.$$

Given $\dot{\theta}^2 \approx 0$, $\theta^2 \approx 0$, thus $\cos\theta \simeq 1$, $\sin\theta \simeq \theta$.

$$3mR^2\ddot{\theta} + mgR\theta = 0$$
$$\ddot{\theta} + \frac{g}{3R}\theta = 0$$
thus $\Rightarrow$ $\omega = \sqrt{\dfrac{g}{3R}}$