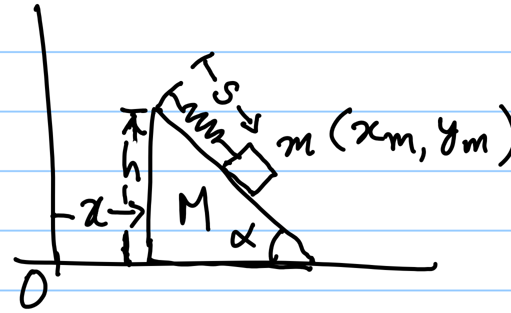
Quiz-I Solutions

1 DOF = 2

Generalized coordinates: (x, s)



$x_m = x + s\cos\alpha$

$y_m = h - s\sin\alpha$

$\dot{x}_m = \dot{x} + \dot{s}\cos\alpha$ & $\dot{y}_m = -\dot{s}\sin\alpha$

$$T = \frac{M}{2}\dot{x}^2 + \frac{m}{2}(\dot{x}_m^2 + \dot{y}_m^2) = \frac{M}{2}\dot{x}^2 + \frac{m}{2}(\dot{x}^2 + \dot{s}^2 + 2\dot{x}\dot{s}\cos\alpha)$$

$$V = mg\,y_m + \frac{k}{2}(s-l)^2 = mg(h - s\sin\alpha) + \frac{k}{2}(s-l)^2$$

So $$L = \frac{M}{2}\dot{x}^2 + \frac{m}{2}(\dot{x}^2 + \dot{s}^2 + 2\dot{x}\dot{s}\cos\alpha) - mg(h - s\sin\alpha) - \frac{k}{2}(s-l)^2$$

For the E-L equations,

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + m\dot{s}\cos\alpha = \text{const}! \quad \text{— (1a)} \qquad \left(\text{Since } \frac{\partial L}{\partial x} = 0\right)$$

Or, $(M+m)\ddot{x} + m\ddot{s}\cos\alpha = 0$ — (1b) $\Rightarrow \ddot{x} = \dfrac{-m\ddot{s}\cos\alpha}{(M+m)}$ — (2)

$\dfrac{\partial L}{\partial \dot{s}} = m(\dot{s} + \dot{x}\cos\alpha)$ | $\dfrac{\partial L}{\partial s} = +mg\sin\alpha - k(s-l)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0 \Rightarrow m(\ddot{s} + \ddot{x}\cos\alpha) - mg\sin\alpha + k(s-l) = 0 \quad \text{— (3)}$$

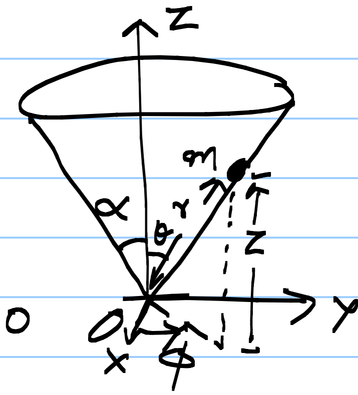
Substituting (2) in (3) $\Rightarrow \ddot{s}\left(m - \dfrac{m^2\cos^2\alpha}{(M+m)}\right) + ks - kl - mg\sin\alpha = 0$ — (4)

Let $q = ks - kl - mg\sin\alpha$ | Also $\mu = m - \dfrac{m^2\cos^2\alpha}{(M+m)}$

$\ddot{q} = k\ddot{s} \Rightarrow \ddot{s} = \ddot{q}/k$

(4) $\Rightarrow \mu\ddot{q}/k + q = 0$ i.e. $\ddot{q} + (k/\mu)q = 0$

So $\underline{\underline{\omega = \sqrt{k/\mu}}}$ where $\underline{\underline{\mu = m\left(1 - \dfrac{m\cos^2\alpha}{M+m}\right)}}$



We shall use sperical polar (r, θ, ϕ), so that $\theta = \alpha$; (r, ϕ) - are the generalized coordinates. DOF = 2.

velocity of m, $\bar{v} = \dot{r}\hat{r} + r\sin\theta\dot{\phi}\hat{\phi}$ | $\dot{\theta} = 0$

So $T = \frac{m}{2}(\bar{v}\cdot\bar{v}) = \frac{m}{2}\left(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2\right)$ | $\theta = \alpha$

$V = mgz = mgr\cos\alpha$

So

$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2\right) - mgr\cos\alpha$ | $\mathcal{L}$ is cyclic in ϕ

$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$ | $\frac{\partial L}{\partial r} = mr\sin^2\alpha\,\dot{\phi}^2 - mg\cos\alpha$

So E-L eq. ① $\Rightarrow m\ddot{r} - mr\sin^2\alpha\,\dot{\phi}^2 + mg\cos\alpha = 0$ — ①

Or, $\ddot{r} - r\sin^2\alpha\,\dot{\phi}^2 + g\cos\alpha = 0$

$p_\theta = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\alpha\,\dot{\phi} = \underline{\underline{\text{conserved}}}$ — ② | $\frac{\partial L}{\partial \phi} = 0$