Lecture 10:
Vector fields, Curl and Divergence

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Vector fields

Definition: A vector field in $\mathbb{R}^n$ is a function $F : \mathbb{R}^n \to \mathbb{R}^n$ that assigns to each $\mathbf{x} \in \mathbb{R}^n$ a vector $F(\mathbf{x})$. A vector field in $\mathbb{R}^n$ with domain $U \subset \mathbb{R}^n$ is called a vector field on U.

Geometrically, a vector field F on U is interpreted as attaching a vector to each point of U. Thus, there is a subtle difference between a vector field in $\mathbb{R}^n$ and a function from $\mathbb{R}^n$ to $\mathbb{R}^n$.

> When a function $F : \mathbb{R}^n \to \mathbb{R}^n$ is viewed as a vector field, for each x the vector $F(\mathbf{x})$ is identified with the vector that starts at the point $\mathbf{x}$ and points to $F(\mathbf{x})$, i.e., $F(\mathbf{x})$ is identified with the vector that is obtained by translating $F(\mathbf{x})$ to the point $\mathbf{x}$.

Thus every vector field on $U \subset \mathbb{R}^n$ is uniquely determined by a function from $U \to \mathbb{R}^n$.

Examples of vector fields

- The gravitational force field describes the force of attraction of the earth on a mass m and is given by

$$\mathbf{F} = -\frac{mMG}{r^3}\mathbf{r},$$

where $\mathbf{r} := (x, y, z)$ and $r := \|\mathbf{r}\|$. The vector field F points to the centre of the earth.

- The vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) := (-y, x)$ is a rotational vector field in $\mathbb{R}^2$ which rotates a vector in the anti-clockwise direction by an angle $\pi/2$.

- Let $\mathbf{x} : [0, 1] \to \mathbb{R}^n$ be C^1 and $\Gamma := \mathbf{x}([0, 1])$. Then $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}^n$ given by $F(\mathbf{x}(t)) = \mathbf{x}'(t)$ is a tangent vector field on Γ.

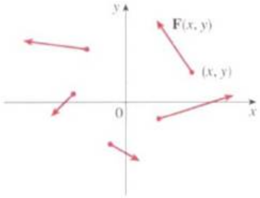

FIGURE 2
Vector field on $\mathbb{R}^2$

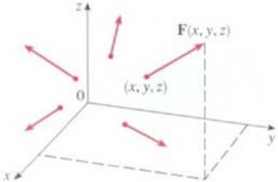

FIGURE 3
Vector field on $\mathbb{R}^3$

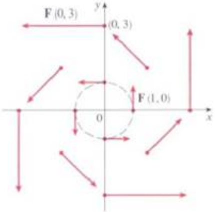

FIGURE 4
$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

Figure: Examples of vector fields

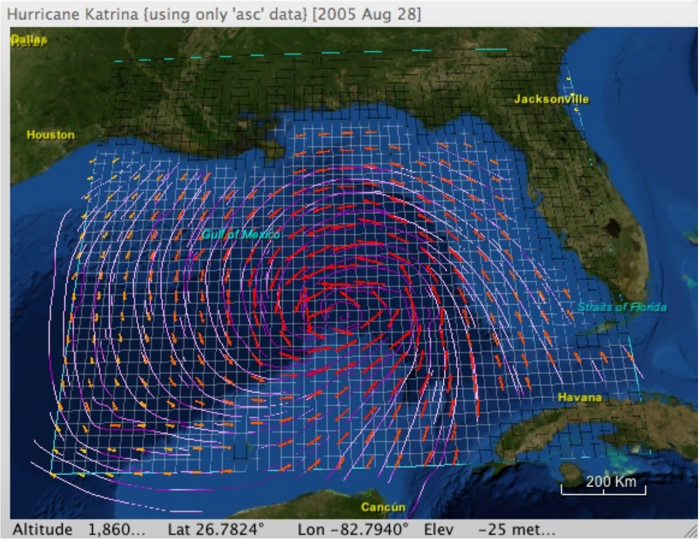

Figure: Vector field representing Hurricane Katrina

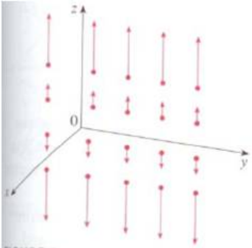

FIGURE 8
$\mathbf{F}(x, y, z) = z\,\mathbf{k}$

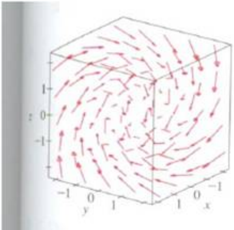

FIGURE 9
$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$

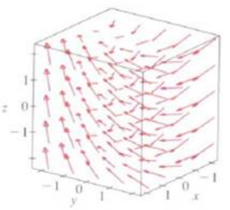

FIGURE 10
$\mathbf{F}(x, y, z) = y\,\mathbf{i} - 2\,\mathbf{j} + x\,\mathbf{k}$

Figure: Examples of vector fields

Scalar fields

A function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is called a scalar field on U as it assigns a number to each point in U.

The temperature of a metal rod that is heated at one end and cooled on another is described by a scalar field $T(x, y, z)$. The flow of heat is described by a vector field. The energy or heat flux vector field is given by $\mathbf{J} := -\kappa \nabla T$, where $\kappa > 0$.

- Let F be a vector field in $\mathbb{R}^n$. Then $F := (f_1, \ldots, f_n)$ for some scalar fields $f_1, \ldots, f_n$ on $\mathbb{R}^n$. We say that F is a C^k vector field if $f_1, \ldots, f_n$ are C^k functions.

All vector fields are assumed to be C^1 unless otherwise noted.

Gradient vector fields

If $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 scalar field then $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field in $\mathbb{R}^n$.

- A vector field F in $\mathbb{R}^n$ is said to be a gradient vector field or a conservative vector field if there is a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ such that $F = \nabla f$. In such a case, f is called a scalar potential of the vector field F.

Obviously not every vector field is a gradient vector field.

Example: The vector field $F(x, y) := (y, -x)$ is not a gradient vector field. Indeed, if $F = \nabla f$ then $f_x = y$ and $f_y = -x$.

Consequently, $f_{xy} = 1$ and $f_{yx} = -1$. Hence F is NOT a C^1 vector field, which is a contradiction.

Integral curves for vector fields

Definition: Let F be a vector field in $\mathbb{R}^n$. Then a C^1 curve $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is said to be an integral curve for the vector field F if $F(\mathbf{x}(t)) = \mathbf{x}'(t)$ for $t \in [a, b]$.

Obviously, F is a tangent (velocity) vector field on the integral curve. Thus integral curves provide a geometric picture of a vector field.

Integral curves are also called flow lines or streamlines.

An integral curve may be viewed as a solution of a system of ODE. Indeed, for $n = 3$ and $F = (P, Q, R)$, we have

$$\begin{aligned} x'(t) &= P(x(t), y(t), z(t)), \\ y'(t) &= Q(x(t), y(t), z(t)), \\ z'(t) &= R(x(t), y(t), z(t)). \end{aligned}$$

Example

Find integral curves for the vector field $F(x, y) = (-y, x)$.

Note that $\mathbf{r}(t) = (\cos t, \sin t)$ is an integral curve for F. The other integral curves are also circles and are of the form

$$\gamma(t) := (r\cos(t - t_0),\ r\sin(t - t_0)).$$

This follows from the system of ODE $x' = -y$ and $y' = x$ which gives $x'' + x = 0$.

Divergence of vector fields

Definition: Let $F = (f_1, \ldots, f_n)$ be a vector field in $\mathbb{R}^n$. Then the divergence of F is a scalar field on $\mathbb{R}^n$ given by

$$\mathrm{div} F = \partial_1 f_1 + \cdots + \partial_n f_n = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}.$$

Define the del operator

$$\nabla := (\partial_1, \ldots, \partial_n) = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}) = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \cdots + \mathbf{e}_n \frac{\partial}{\partial x_n}.$$

Then applying ∇ to a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ we obtain the gradient vector field

$$\nabla f = (\partial_1 f, \ldots, \partial_n f).$$

Divergence of vector fields

Taking dot product of ∇ with a vector field $F = (f_1, \ldots, f_n)$ we obtain the divergence

$$\nabla \bullet F = \partial_1 f_1 + \cdots + \partial_n f_n = \mathrm{div} F.$$

Physical interpretation: If F represents velocity field of a gas (or fluid) then $\mathrm{div} F$ represents the rate of expansion per unit volume under the flow of the gas (or fluid).

The divergence of $F = (x^2 y, z, xyz)$ is given by

$$\mathrm{div} F = 2xy + 0 + xy = 3xy.$$

Examples

- The streamlines of the vector field $F(x, y) := (x, y)$ are straight lines directed away from the origin.

  For fluid flow, this means the fluid is expanding as it moves out from the origin, so $\mathrm{div} F$ should be positive. Indeed, we have

$$\mathrm{div} F = 2 > 0.$$

- Next, consider the vector field $F(x, y) := (x, -y)$. Then

$$\mathrm{div} F = 0$$

  so that neither expansion nor compression takes place.

Cross product in $\mathbb{R}^3$

The cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ denoted by $\mathbf{u} \times \mathbf{v}$ is the unique vector satisfying the following conditions:

- $\mathbf{u} \perp (\mathbf{u} \times \mathbf{v})$ and $\mathbf{v} \perp (\mathbf{u} \times \mathbf{v})$.
- $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$, where $0 \le \theta \le \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
- $\det(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}) \ge 0$, where $[\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}]$ is the matrix whose columns are $\mathbf{u}, \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$.

Cross product in $\mathbb{R}^3$

It is customary to denote the standard basis in $\mathbb{R}^3$ by

$$\mathbf{i} := (1, 0, 0), \ \ \mathbf{j} := (0, 1, 0) \ \text{ and } \ \mathbf{k} := (0, 0, 1).$$

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ then

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_2 v_3 - v_2 u_3)\mathbf{i} + (u_1 v_3 - v_1 u_3)\mathbf{j} + (u_1 v_2 - v_2 u_1)\mathbf{k} \\ &= \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{aligned}$$

the last equality is only symbolic.

Curl of vector fields in $\mathbb{R}^3$

The curl of a vector field $F = (f_1, f_2, f_3)$ in $\mathbb{R}^3$ is a vector field in $\mathbb{R}^3$ and is given by

$$\mathrm{curl} F = \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_1 & \partial_2 & \partial_3 \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}.$$

Thus curl of F is obtained by taking cross product of the del operator $\nabla = \dfrac{\partial}{\partial x}\mathbf{i} + \dfrac{\partial}{\partial y}\mathbf{j} + \dfrac{\partial}{\partial z}\mathbf{k}$ with $F = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$.

Curl of a vector field represents rotational motion when, for example, the vector field represents flow of a fluid.

Irrotational vector field

A vector field F in $\mathbb{R}^3$ is called irrotational if $\mathrm{curl} F = 0$. This means, in the case of a fluid flow, that the flow is free from rotational motion, i.e, no whirlpool.

Fact: If f be a C^2 scalar field in $\mathbb{R}^3$. Then ∇f is an irrotational vector field, i.e., $\mathrm{curl}(\nabla f) = 0$.

Proof: We have $\mathrm{curl}(\nabla f) = \nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = 0$ because of the equality of mixed partial derivatives.

Observation: If $\mathrm{curl} F \neq 0$ then F is not a conservative (gradient) vector field.

Examples

- Let $F(x, y, z) := (xy, -\sin z, 1)$. Then
$$\mathrm{curl} F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -\sin z & 1 \end{vmatrix} = (\cos z, 0, -x).$$

- Let $F(x, y, z) := (y, -x, 0)$. Then $\mathrm{curl} F = (0, 0, -2)$ and so F is not a conservative vector field.

- Let $F(x, y, z) := (y\mathbf{i} - x\mathbf{j})/(x^2 + y^2)$. Then $\mathrm{curl} F = 0$ and hence F is irrotational. However, F is NOT a conservative vector field, that is, $F \neq \nabla f$ for some scalar potential f.

Scalar curl

Let $F = (P, Q)$ be a vector field in $\mathbb{R}^2$. Then identifying $\mathbb{R}^2$ with the x-y plane in $\mathbb{R}^3$, F can be identified with the vector field $F = P\mathbf{i} + Q\mathbf{j}$ in $\mathbb{R}^3$. Then we have

$$\mathrm{curl} F = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k}.$$

The scalar field $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is called the scalar curl of the vector field $F = (P, Q)$.

The scalar curl of $F(x, y) := (-y^2, x)$ is given by

$$\partial_x(x) - \partial_y(-y^2) = 1 + 2y.$$

Divergence of curl

Fact: Let F be a C^2 vector field in $\mathbb{R}^3$. Then

$$\mathrm{div}(\mathrm{curl}F) = \nabla \bullet \nabla \times F = 0.$$

Proof: Since

$$\mathrm{curl}F = (\partial_y f_3 - \partial_z f_2)\mathbf{i} + (\partial_z f_1 - \partial_x f_3)\mathbf{j} + (\partial_x f_2 - \partial_y f_1),$$

$\mathrm{div}(\mathrm{curl}F) = 0$ because the mixed partial derivatives are equal.

A vector field F in $\mathbb{R}^3$ is said to have vector potential if there exists a vector field G in $\mathbb{R}^3$ such that $F = \mathrm{curl}G$.

Observation: If F has a vector potential then $\mathrm{div}(F) = 0$.

Laplace operator

Example: If $F(x, y, z) := (x, y, z)$ then $\mathrm{div} F = 3 \neq 0$ so F does not have a vector potential.

Laplace operator ∇^2: Let f be a C^2 scalar field in $\mathbb{R}^n$. Then

$$\nabla^2 f := \nabla \bullet \nabla f = \mathrm{div}(\nabla f) = \partial_1^2 f + \cdots + \partial_n^2 f$$

defines the Laplace operator ∇^2 on f.

For a C^2 function $u : \mathbb{R}^3 \to \mathbb{R}$ the partial differential equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is called Laplace equation.

Summary

Let F be a C^2 vector field in $\mathbb{R}^3$ and $u : \mathbb{R}^3 \to \mathbb{R}$ be a C^2 scalar field. Then

- $\mathrm{div}F = \nabla \bullet F$ and $\mathrm{curl}F = \nabla \times F$,
- $\mathrm{div}(\nabla u) = \nabla^2 u$ and $\mathrm{curl}(\nabla u) = \nabla \times \nabla u = 0$,
- $\mathrm{div}(\mathrm{curl}F) = \nabla \bullet \nabla \times F = 0$.

*** End ***