Lecture 5: Differentiability

Rafikul Alam
Department of Mathematics
IIT Guwahati

Differential Calculus for $f : \mathbb{R}^n \to \mathbb{R}$

Question: Let $f : \mathbb{R}^n \to \mathbb{R}$. What does it mean to say that f is differentiable?

Task: Define differentiability of f at $\mathbf{a} \in \mathbb{R}^n$ and determine the derivative $\mathrm{D}f(\mathbf{a})$.

Wish List:

- f is differentiable at $\mathbf{a} \Rightarrow f$ is continuous at $\mathbf{a}$.
- Sum, product and chain rules hold for $\mathrm{D}f(\mathbf{a})$.
- Mean Value Theorem and Taylor's Theorem hold for f.

Differentiability of $f : (c, d) \subset \mathbb{R} \to \mathbb{R}$

- Conventional: f is differentiable at $a \in (c, d)$ if there exists $\alpha \in \mathbb{R}$ such that

$$\alpha = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- Clever: f is differentiable at $a \in (c, d)$ if there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \alpha h|}{|h|} = 0.$$

- Smart: f is differentiable at $a \in (c, d)$ if there exists a linear map $L : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - L(h)|}{|h|} = 0.$$

Differentiability of $f : U \subset \mathbb{R}^n \to \mathbb{R}$

Smart: Let $U \subset \mathbb{R}^n$ be open. Then $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a} \in U$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{\mathbf{h} \to 0} \frac{|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - L(\mathbf{h})|}{\|\mathbf{h}\|} = 0.$$

The linear map L is called the derivative of f at $\mathbf{a}$ and is denoted by $\mathrm{D}f(\mathbf{a})$, that is, $L = \mathrm{D}f(\mathbf{a})$.

Fact: If $L : \mathbb{R}^n \to \mathbb{R}$ is linear then $L(\mathbf{x}) = \mathbf{p} \bullet \mathbf{x} = \langle \mathbf{x}, \mathbf{p} \rangle$ for some $\mathbf{p} \in \mathbb{R}^n$.

Theorem: If f is differentiable at $\mathbf{a} \in \mathbb{R}^n$ then $\nabla f(\mathbf{a})$ exists and the derivative $\mathrm{D}f(\mathbf{a}) : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\mathrm{D}f(\mathbf{a})\mathbf{h} = \nabla f(\mathbf{a}) \bullet \mathbf{h} = \langle \mathbf{h}, \nabla f(\mathbf{a}) \rangle.$$

- Conventional: $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$ if $\nabla f(\mathbf{a})$ exists and

$$\lim_{\mathbf{h} \to 0} \frac{|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \bullet \mathbf{h}|}{\|\mathbf{h}\|} = 0.$$

- Clever: $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$ if there exists $\alpha \in \mathbb{R}^n$ such that

$$\lim_{\mathbf{h} \to 0} \frac{|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \alpha \bullet \mathbf{h}|}{\|\mathbf{h}\|} = 0.$$

Examples

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) = 0$ and $f(x,y) := xy\dfrac{x^2 - y^2}{x^2 + y^2}$ if $(x,y) \neq (0,0)$. Then

- f is continuous at $(0,0)$ and $\nabla f(0,0) = (0,0)$.
- Now

$$\frac{|f(h,k) - f(0,0) - \nabla f(0,0) \bullet (h,k)|}{\|(h,k)\|} \leq \frac{|hk|}{\|(h,k)\|} \to 0$$

  Hence f is differentiable at $(0,0)$.

Consider $g : \mathbb{R}^3 \to \mathbb{R}$ given by $g(x,y,z) := 3x + 5y - z$. Then g is differentiable. Find $\mathrm{D}g(x,y,z)$.

Affine approximation

Define the error function $e : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by

$$e(\mathbf{h}) := \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \bullet \mathbf{h}}{\|\mathbf{h}\|}.$$

- Then f is differentiable at $\mathbf{a}$ if and only if

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \bullet \mathbf{h} + e(\mathbf{h})\|\mathbf{h}\|$$

  and $e(\mathbf{h}) \to 0$ as $\|\mathbf{h}\| \to 0$.

- The affine function $y = f(\mathbf{a}) + \nabla f(\mathbf{a}) \bullet \mathbf{h}$ approximates $f(\mathbf{a}+\mathbf{h})$ for small $\|\mathbf{h}\| \iff f$ is differentiable at $\mathbf{a}$.

Geometric interpretation

Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a} \in \mathbb{R}^n$. Then

$$y = f(\mathbf{a}) + \nabla f(\mathbf{a}) \bullet \mathbf{x}$$

represents

- For $n = 1$: a line $y = f(a) + f'(a)x$ passing through $(0, f(a)) \in \mathbb{R}^2$ that approximates $f(a + x)$.
- For $n = 2$: a plane $z = f(a, b) + f_x(a, b)x + f_y(a, b)y$ passing through $(0, 0, f(a, b)) \in \mathbb{R}^3$ that approximates $f(a + x, b + y)$.
- For $n \geq 3$: a hyperplane $y = f(\mathbf{a}) + \partial_1 f(\mathbf{a})x_1 + \cdots + \partial_n f(\mathbf{a})x_n$ passing through $(\mathbf{0}, f(\mathbf{a})) \in \mathbb{R}^{n+1}$ that approximates $f(\mathbf{a} + \mathbf{x})$.

Implications of differentiability

Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$.

- If f is differentiable at $\mathbf{a}$ then f is continuous at $\mathbf{a}$.
- If f is differentiable at $\mathbf{a}$ then directional derivatives exist for all $\mathbf{u} \in \mathbb{R}^n$ and

$$\mathrm{D}_{\mathbf{u}} f(\mathbf{a}) = \mathrm{D}f(\mathbf{a})\mathbf{u} = \nabla f(\mathbf{a}) \bullet \mathbf{u}.$$

Proof: Use

$$f(\mathbf{a} + \mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \bullet \mathbf{h} + e(\mathbf{h})\|\mathbf{h}\|$$

and the fact that $e(\mathbf{h}) \to 0$ as $\|\mathbf{h}\| \to 0$.

Example

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) = 0$ and $f(x,y) := \dfrac{x^2 y}{x^4 + y^2}$ if $(x,y) \neq (0,0)$. Then

- f is NOT continuous at $(0,0) \Rightarrow f$ is not differentiable at $(0,0)$.
- $\mathrm{D}_{\mathbf{u}} f(0,0)$ exists for all $\mathbf{u} \in \mathbb{R}^2$ and $\nabla f(0,0) = (0,0)$.
- For $\mathbf{u}$ such that $u_1 u_2 \neq 0$, we have

$$\mathrm{D}_{\mathbf{u}} f(0,0) = u_1^2 / u_2 \neq \nabla f(0,0) \bullet \mathbf{u}.$$

Moral: The equality $\mathrm{D}_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \bullet \mathbf{u}$ may not hold if f is NOT differentiable at $\mathbf{a}$.

Properties of derivative

Fact: Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a} \in \mathbb{R}^n$. Then

- $\mathrm{D}(f + g)(\mathbf{a}) = \mathrm{D}f(\mathbf{a}) + \mathrm{D}g(\mathbf{a})$.
- $\mathrm{D}(fg)(\mathbf{a}) = \mathrm{D}f(\mathbf{a})g(\mathbf{a}) + f(\mathbf{a})\mathrm{D}g(\mathbf{a})$.

Proof: Use

$$f(\mathbf{a} + \mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \bullet \mathbf{h} + e(\mathbf{h})\|\mathbf{h}\|$$

and the fact that $e(\mathbf{h}) \to 0$ as $\|\mathbf{h}\| \to 0$.

Sufficient condition for differentiability

Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$. If $\partial_i f(\mathbf{x})$ exists for $i = 1, 2, \ldots, n$, and are continuous on $B(\mathbf{a}, \epsilon)$ for some $\epsilon > 0$, then f is differentiable at $\mathbf{a}$.

Proof: Use MVT for partial derivatives.

Remark: f differentiable at $\mathbf{a} \not\Rightarrow \partial_i f(\mathbf{x})$ is continuous at $\mathbf{a}$.

Example: Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) = 0$ and $f(x, y) := (x^2 + y^2)\sin(1/(x^2 + y^2))$ if $(x, y) \neq (0, 0)$.

*** End ***