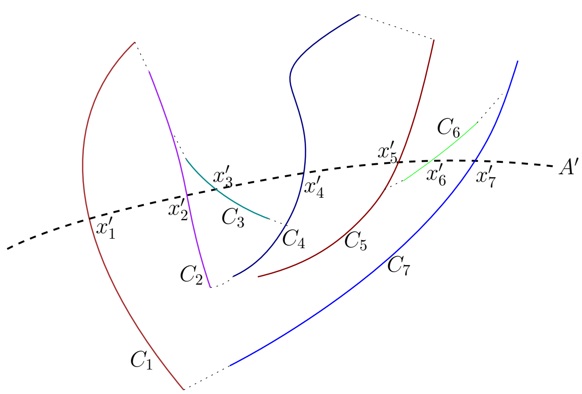
- A subset of a poset such that every two elements of this subset are comparable is called a *chain*. A *maximal chain* is a chain that is not a proper subset of any other chain. A *maximum chain* is a chain that has cardinality at least as large as every other chain. The *height of a poset* is the cardinality of a maximum chain.

- A subset of a poset is called an *antichain* if every two elements of this subset are not comparable. A *maximal antichain* is an antichain that is not a proper subset of any other antichain. A *maximum antichain* is an antichain that has cardinality at least as large as every other antichain. The *width of a poset* is the cardinality of a maximum antichain.

- *Dilworth's theorem*: If w is the width of a finite non-empty poset $(S, \preccurlyeq)$, then there is a partition of elements in S into w chains.

  Proof is by induction on the cardinality of S:

  * Basis: Consider a set S with one element, say $S = \{x\}$. In $(S, \preccurlyeq)$, the only maximum antichain is $\{x\}$, its size is 1, and $C_1 = \{x\}$ with $C_1 = S$.
  * Induction hypothesis: Let x be a maximal element of $(S, \preccurlyeq)$. And, let S' be $S - \{x\}$. For every set $S'' \subseteq S'$, if w'' is the width of $(S'', \preccurlyeq)$, then there is a partition of elements in S'' into w'' chains.



Illustrating a maximum antichain A' comprising elements $x'_1, x'_2, \ldots, x'_7$ with a black dashed line. Also, shown a partitioning of $S' = S - \{a\}$ into seven chains ($\mathcal{C} = \{C_1, C_2, \ldots C_7\}$), each is in a different color.

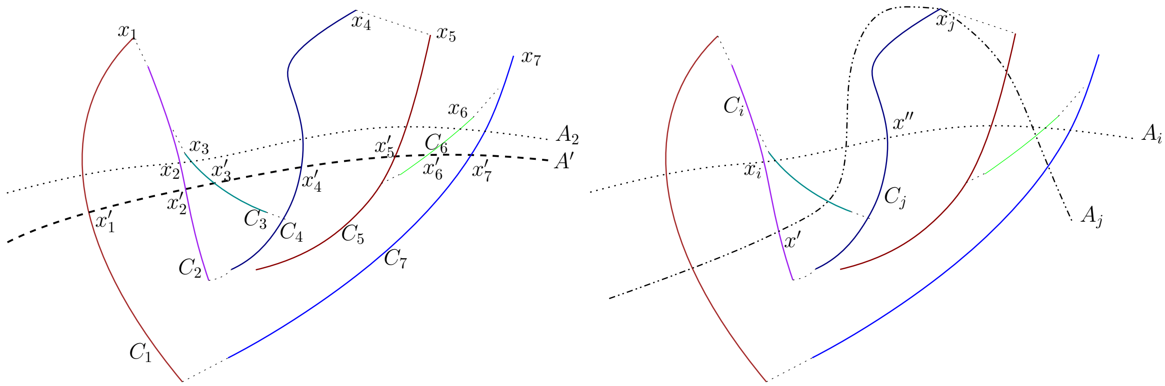
Let A' be a maximum antichain of poset $(S', \preccurlyeq)$ comprising w' elements. Also, let $\mathcal{C} = \{C_1, C_2, \ldots, C_{w'}\}$ be the chain partitioning of elements in S' due to induction hypothesis.

- Lemma 1: For every $i \in [1, w']$, $|C_i \cap A'| = 1$.

  Proof: Suppose $C_j \cap A' = \phi$ for $C_j \in \mathcal{C}$. This implies, from the pigeonhole principle, there exists a chain $C_{j'} \in \mathcal{C}$ such that $C_{j'}$ contributes at least two elements, say y' and y'', to A'. However, since y' and y'' are comparable, A' is not an antichain. □

- Lemma 2: For $i = 1, 2, \ldots, w'$, let x_i be the maximal element in C_i that belongs to a maximum antichain A_i of $(S', \preccurlyeq)$. Then, $A = \{x_1, x_2, \ldots, x_{w'}\}$ is an antichain of $(S', \preccurlyeq)$.

  Proof: For every i, A_i always exists, since an element a'_i of C_i belongs to antichain A'. (For example, a_i can be found by walking along C_i from top to bottom.) Refer to left figure.



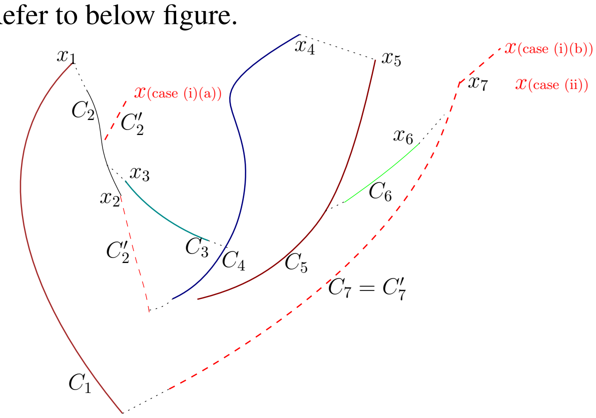
On the left, illustrating $x_1, x_2, \ldots, x_7$ are the maximal elements along each of the chains in $\mathcal{C} = \{C_1, C_2, \ldots C_7\}$ so that there is a maximum antichain A_i that contains x_i for every $i \in [1, 7]$. For example, maximum antichain A_2 has x_2 whereas no element above x_2 along C_2 can belong to a maximum antichain. Noting that, for every $i \in [1, 7]$, x'_i of A' is $\preccurlyeq x_i$, it is guaranteed for x_i to exist on C_i.
On the right figure, illustrating maximum antichains A_i and A_j, respectively containing x_i and x_j.

Suppose $x_i \preccurlyeq x_j$. Let $x' \in C_i \cap A_j$. Then, since $x' \preccurlyeq x_i$, from transitivity, $x' \preccurlyeq x_j$, leading to a contadiction of A_j being an antichain. Analogously, suppose $x_j \preccurlyeq x_i$. Let $x'' \in C_j \cap A_i$. Then, since $x'' \preccurlyeq x_j$, from transitivity, $x'' \preccurlyeq x_i$, leading to a contadiction of A_i being an antichain. Refer to right figure.

For every $x_i, x_j \in A$, since $x_j \not\preccurlyeq x_i$ and $x_i \not\preccurlyeq x_j$, A is an antichain. □

* Induction step: Consider the poset $(S, \preccurlyeq)$. Since x is a maximal element of this poset, there are two possibilities: (i) $x_i \preccurlyeq x$ for some $i \in [1, \ldots, w']$, or (ii) $x_i \not\preccurlyeq x$ for every $i \in [1, \ldots, w']$. Note that w' is the width of $(S', \preccurlyeq)$ and S' is partitioned into w' chains.

  In Case (i), consider $(S' - C'_i, \preccurlyeq)$, where C'_i is the chain underneath x_i (including x_i) in $(S', \preccurlyeq)$. Since no element in $C_i - C'_i$ belongs to a maximum antichain, the width of $(S' - C'_i, \preccurlyeq)$ is $w' - 1$. Since $S' - C'_i \subset S$, from the induction hypothesis (by the means of strong induction), $S' - C'_i$ is partitioned into $w' - 1$ chains. Let $\mathcal{C}'$ be this set of $w' - 1$ chains. By including x, S can be partitioned into w' chains, and the width of the resultant poset is w'. (That is, the width of $(S, \preccurlyeq)$ does not change by including x into S'.) Refer to below figure.



Illustrating Case (i)(a) wherein x is above x_2, Case (i)(b) wherein x is above x_7, and Case (ii) wherein x is not related to any x_i for $i \in [1, \ldots, 7]$. In both the Cases (i)(a) and (i)(b), elements along the red colored dashed line are removed before applying induction hypothesis to the rest of the poset. Significantly, due to the choice of x_2, no element in $C_2 - C'_2$ belongs to a maximum antichain.

In Case (ii), applying induction hypothesis to $(S', \preccurlyeq)$ leads to partitioning S' into w' chains $C_1, C_2, \ldots$, $C_{w'}$. These with a maximal chain comprising x is a partition of S into $w' + 1$ chains. Further, due to Lemma 2 and since no x_i is related to x, $A' \cup \{x\}$ is a maximum antichain of size $w' + 1$. Refer to above figure. $\square$