

- Problem: Given a set P of n points in the plane, cover them by a minimum number of disks, each of unit radius.
- Observation 1: For a unit disk D that covers points in set $Q (\subseteq P)$, it is always possible to translate D so that the translated disk D' continue to cover points in Q and the $bd(D')$ contains (at least) two points of Q or the topmost point on the $bd(D')$ is a point in Q . We call the former a type-i disk and the latter a type-ii disk.

The total number of pairwise distinct type-i and type-ii disks possible with points in Q is at most $2 \binom{|Q|}{2} + |Q| = |Q|^2$.

A naive algorithm: For $1 \leq i \leq k$, choose every possible set of i disks from these $|Q|^2$ disks and check whether these together cover points in Q .

This algorithm takes $\sum_{i=1}^k O(\binom{|Q|^2}{i}) \cdot O(|Q|^i)$, i.e., $O(|Q|^{2k}|Q|k)$, i.e., $O(k|Q|^{2k+1})$ time. ——— (1a)

- Consider a grid G that contains points in P and each cell with width Δ .

Observation 2: Any $\Delta \times \Delta$ grid cell C contained anywhere in a grid cell $(\Delta + 1) \times (\Delta + 1)$ can be covered by $(\Delta + 1)^2$ unit disks. [Proof: $(\Delta + 1)^2$ unit squares together can cover C , and there is a unit disk containing a unit square.]

From Observation 1 and (1a), the cover of points in C with type-i and type-ii disks can be computed in $O((\Delta + 1)^2 |C|^{2(\Delta+1)^2+1})$ time. ——— (2a)

- A deterministic algorithm:

- assume points in P are located in the first quadrant of the coordinate system
- let G be a grid with cells of width $\Delta \times \Delta$ such that every point in P belongs to a cell of G and the southwest corner of the southwest grid cell of G is located at the origin of the coordinate system

(i) for each grid cell C of G containing $Q (\subseteq P)$ points with $|Q| \geq 1$

(ii) for $i \leftarrow 1$ to $(\Delta + 1)^2$

(iii) for every i -combination R of $O(|Q|^2)$ type-i and type-ii disks

(iv) if disks in R together cover points in C then save R in F_C and break from (ii)

(v) return $F = \cup_C F_C$

It is obvious that F_C is an optimal solution for covering points in C . And, hence F is a feasible solution to cover points in P .

As there are n non-empty grid cells, from (2a), the time complexity is $O((\Delta + 1)^2 n^{2(\Delta+1)^2+2})$. By choosing $\Delta = \frac{12}{\epsilon}$, it becomes $O(\frac{1}{\epsilon^2} n^{O(1/\epsilon^2)})$, i.e., $O(n^{\lg n} \frac{1}{\epsilon^2} n^{O(1/\epsilon^2)})$, i.e., $n^{O(1/\epsilon^2)}$.

- Let $O = \{D_1, \dots, D_{OPT}\}$ be an optimal solution to cover points in P . For each grid cell C , let O_C be the set of disks in O that intersect C .

Observation 3: For every grid cell C , since $|F_C| \leq |O_C|$, $|F| \leq \sum_C |O_C|$.

However, a disk $D_i \in O$ can intersect at most four cells of the grid, and as such it can appear in $\cup_C O_C$ at most four times. Therefore, $|F| \leq 4|O|$.

Considering $D_i \in O$ will appear in $\cup_C O_C$ more than once iff D_i is not fully contained in a grid cell of G (and a disk may not intersect all the four cells), we will be interested in the expected analysis for the approx factor . . .

- A randomized algorithm:

Randomly shift a grid G by a position vector \vec{r} randomly chosen from $[0, \Delta) \times [0, \Delta)$ while the new grid G' covers points in P . (Hence, we had considered earlier a $\Delta \times \Delta$ grid being positioned anywhere in a $(\Delta + 1) \times (\Delta + 1)$ cell.) And then run through Steps (i)-(v) of the deterministic algorithm stated above.

- This grid shifting is popularly known as the *Hochbaum-Maass shifting* technique.
- The probability that a unit disk D is not fully contained in a grid cell of G is equal to probability the endpoints of D 's projection along the x -axis are in different cells on x -axis or the endpoints of D 's projection along the y -axis are in different cells on y -axis, which is at most $\frac{4}{\Delta}$. [Proof: projected endpoints p', p'' along the x -axis (resp. y -axis) are in different cells iff $xcoord(\vec{r}) \in [p', p'']$.]
- For every i , let X_i be an indicator variable that is 1 iff D_i (a disk in O) is not fully contained in a single cell of G' . Then,

$$\begin{aligned}
 E[F] & \\
 & \leq E[OPT + \sum_{i=1}^{OPT} 3X_i] \quad (\text{as a disk can intersect with at most 4 grid cells}) \\
 & = OPT + 3E[\sum_{i=1}^{OPT} X_i] \\
 & = OPT + 3 \sum_{i=1}^{OPT} E[X_i] \\
 & = OPT + 3 \sum_{i=1}^{OPT} pr[X_i = 1] \\
 & \leq OPT + 3 \sum_{i=1}^{OPT} \frac{4}{\Delta} \\
 & = (1 + \frac{12}{\Delta})OPT \\
 & = (1 + \epsilon)OPT \quad (\text{hence, we had chosen } \Delta = \frac{12}{\epsilon})
 \end{aligned}$$

- As a whole, when a grid with $\frac{12}{\epsilon}$ cell width is chosen, this randomized approximation algorithm computes a disk cover of expected size $(1 + \epsilon)OPT$ in $n^{O(1/\epsilon^2)}$ time.

References:

- Approximation schemes for covering and packing problems in image processing and VLSI, D. S. Hochbaum and W. Maass, JACM, 32(01): 130-136, 1985.
- Geometric Approximation Algorithms by S. Har-Peled.