Principle: There is no infinite sequence of strictly decreasing non-negative integers.

To prove a proposition P is true, on the way of arriving to a contradiction, suppose $\neg P$ is true. Construct a solution S_1 to $\neg P$, prove the existence of S_1 implies another solution S_2 to $\neg P$, ..., wherein $S_1, S_2, \ldots$ is a decremental sequence of non-negative integers. As you might have noted, this proof method is closely related to proving using well-ordering principle.

- For a positive integer k, $\sqrt{k}$ is irrational if it is not an integer.

  $\sqrt{k} = \frac{m}{n} = \frac{m(\sqrt{k} - \lfloor\sqrt{k}\rfloor)}{n(\sqrt{k} - \lfloor\sqrt{k}\rfloor)} = \frac{m'}{n'}$; since $0 < (\sqrt{k} - \lfloor\sqrt{k}\rfloor) < 1$, $m' < m$, $n' < n$, and m', n' are positive

  further, $\frac{m'}{n'} = \frac{m(\sqrt{k} - \lfloor\sqrt{k}\rfloor)}{n(\sqrt{k} - \lfloor\sqrt{k}\rfloor)} = \frac{nk - m\lfloor\sqrt{k}\rfloor}{m - n\lfloor\sqrt{k}\rfloor}$; hence, m' and n' are non-negative integers

- No integral solutions to $a^2 + b^2 = 3(s^2 + t^2)$, other than the trivial solution $a = b = s = t = 0$.

  noting that $(3 | (a_1^2 + b_1^2))$ iff $(3 | a_1$ and $3 | b_1)$,

  - proof of $\Leftarrow$ is immediate; for $\Rightarrow$, consider the contrapositive: $(((3 \not| a) \vee (3 \not| b)) \Rightarrow (3 \not| (a^2 + b^2))) \equiv ((3 \not| a) \Rightarrow (3 \not| (a^2 + b^2))) \wedge ((3 \not| b) \Rightarrow (3 \not| (a^2 + b^2)))$
  - in proving the former, $(3 \not| a) \Rightarrow (a = 3i + r)$ where $r \in \{1, 2\}$ and $b = 3j + r'$ where $r' \in \{0, 1, 2\}$; the other part's proof is symmetric

  if a_1, b_1, s_1, t_1 is a solution then $s_1, t_1, \frac{a_1}{3}, \frac{b_1}{3}$ is also a solution; further, if $s_1, t_1, \frac{a_1}{3}, \frac{b_1}{3}$ is a solution, then $\frac{a_1}{3}, \frac{b_1}{3}, \frac{s_1}{3}, \frac{t_1}{3}$ is also a solution; hence, leading to a strictly smaller (lexicographically) solution vector

- No non-trivial non-negative integral solution to $x^3 + 2y^3 = 4z^3$.

  - due to each term's power, it is immediate that every possible solution must be non-negative
  - for any solution (x_1, y_1, z_1), considering x_1^3 must be even, x_1 must be even

    pluggin in $x_1 = 2x_2$ leads to $8x_2^3 + 2y_1^3 = 4z_1^3$, which is $4x_2^3 + y_1^3 = 2z_1^3$

    now since y_1 must be even, substituting $y_1 = 2y_2$ leads to $2x_2^3 + 4y_2^3 = z_1^3$

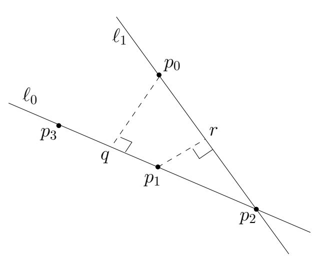
    again, since z_1 must be even, substituting $z_1 = 2z_2$ leads to $x_2^3 + 4y_2^3 = z_2^3$
  - that is, (x_1, y_1, z_1) is a solution implies $(x_2 = \frac{x_1}{2}, y_2 = \frac{y_1}{2}, z_2 = \frac{z_1}{2})$ is a solution implies ... implies $(x_n = \frac{x_1}{2^n}, y_n = \frac{y_1}{2^n}, z_n = \frac{z_1}{2^n})$ is a solution ...

    the infinite descent is due to $x_1 > x_2 > \ldots > x_n > \ldots$; the same is with the y_is and z_is

- *Sylvester-Gallai theorem*: If there are n ($n \geq 3$) points on the plane such that not all on a line, then there exists a line passing through exactly two of these points.

  - let L be the set comprising all lines that pass through at least two points of S; among all pairs (p, ℓ) with $p \in S$ not on $\ell \in L$, choose a pair (p_0, ℓ_0) such that p_0 has the smallest distance to ℓ_0; then ℓ_0 must have exactly two points

    otherwise, the length of $p_1 r$ is lesser to $p_0 q$: from similar triangles, $p_1 p_2 r$ and $p_0 p_2 q$, $\frac{p_0 p_2}{p_1 p_2} = \frac{p_0 q}{p_1 r} \Rightarrow p_1 r = (p_0 q) \frac{p_1 p_2}{p_0 p_2} < p_0 q$



[1]a.k.a., Fermat's method of descent