

- Given a set of n points, any subset of k points that can be separated from the other $n - k$ points by a hyperplane is called a k -set.

For the planar case, the upper bound on the number of k -sets in a set of n points is $O(nk^{1/3})$. [Dey '98]

- First note the following: For a given set of n points in the plane in general position, a pair of points $p, q \in P$ form a k -set if there are exactly k points in the (closed) halfplane below the line passing through p and q .
- Consider the graph $G(P, E)$ that has an edge for every k -set edges of G can be decomposed into $k - 1$ (resp. $n - k + 1$) convex chains
any crossing of two edges of G is an intersection point of one convex chain of C_1, \dots, C_{k-1} with a concave chain of D_1, \dots, D_{n-k+1} ;
hence, there are at most $2(k - 1)(n - k + 1)$ crossings of $G \Rightarrow m^3/n^2 = O(nk)$, which implies $m = O(nk^{1/3})$

The maximum number of k -sets in a set of n points located in \mathbb{R}^d is still unknown.

- The k -level in an arrangement $A(H)$ induced by a set H of n hyperplanes is defined as the set of points with at most $k - 1$ hyperplanes strictly above it, and at most $n - k$ hyperplanes strictly below it.

For the planar case, the upper bound on the complexity of any k -level in an arrangement of n lines is $O(nk^{1/3})$.

- since the k -level problem is the k -set problem in the dual setting

The tight bounds on the maximum complexity of any k -level is still unknown, even in the planar case.

- Assuming lines are in general position, the number of vertices of level at most k in an arrangement $A(H)$ of set H of n lines in the plane is $O(nk)$.

- * choose a subset $R \subseteq H$ at random, by including each line $h \in H$ into R with probability p ; let f denote the number of vertices of level 0 in the arrangement $A(R)$; $E[f] \leq E[|R|] = pn$
- * for any vertex v of $A(H)$, let A_v be an event of v becoming one of the vertices of level 0 in $A(R)$; $p[A_v] = p^2(1 - p)^{\ell(v)}$, where $\ell(v)$ denotes the level of v in $A(H)$;
- * let $V_{\leq k}$ (resp. V) be the set of vertices of level at most k (resp. 0) in $A(H)$;
 $np \geq E[f] = \sum_{v \in V} \text{prob}(A_v) \geq \sum_{v \in V_{\leq k}} \text{prob}(A_v) = \sum_{v \in V_{\leq k}} p^2(1 - p)^{\ell(v)} \geq |V_{\leq k}| p^2(1 - p)^k \Rightarrow |V_{\leq k}| \leq \frac{n}{p(1-p)^k}$;
choosing $p = \frac{1}{k+1}$ to (approximately) maximize RHS gives $|V_{\leq k}| \leq 3(k+1)n$

Clarkson-Shor theorem: The number of vertices of level at most k in an arrangement of n hyperplanes in \mathbb{R}^d is $O(n^{\lfloor d/2 \rfloor} (k+1)^{\lceil d/2 \rceil})$.

— proof is a generalization of the above argument

- Given a set of n points in the plane, any subset of at most k points that can be separated from all other point (there are at least $n - k$ of them) by a line is called a $(\leq k)$ -set.

In the plane, the maximum number of $(\leq k)$ -sets is $\Theta(nk)$, and it is $\Theta(n^{\lfloor d/2 \rfloor} (k+1)^{\lceil d/2 \rceil})$ in \mathbb{R}^d .

References:

- Lectures on Discrete Geometry by J. Matousek.