This proves the exististence of an object that has certain properties.

In this lecture, only three techniques with examples from graph theory are introduced

- *Basic counting argument*: Construct a sample space $\mathcal{S}$ of objects in which the probability is positive that a randomly selected object from $\mathcal{S}$ has the desired properties.

- If $\binom{n}{k}2^{-\binom{k}{2}+1} < 1$, then it is possible to color the edges of K_n with two colors so that it has no monochromatic K_k subgraph.

  proof: prob that a k-clique is monochromatic is $2^{-\binom{k}{2}+1}$; the prob that none of the $\binom{n}{k}$ k-cliques are monochromatic $\geq 1 - \binom{n}{k}2^{-\binom{k}{2}+1}$[1] > 0

- *Expectation argument*: For a discrete random variable X, $p(X \geq E[X]) > 0$ and $p(X \leq E[X]) > 0$.

- Let $G(V, E)$ be an undirected simple graph. Then there is a cut with value at least $|E|/2$.

  proof: for a random bipartition (A, B) of V, for any edge e_i and the corresp. X_i set X_i to 1 if i connects A to B, otherwise, X_i is 0; then the expected value is $\frac{|E|}{2}$

- Given a set of m clauses, let k_i be the number of literals in the i^{th} clause for $i = 1, \ldots, m$. Let $k = min_i k_i$. Then there is a truth assignment that satisfies at least $m(1 - 2^{-k})$ clauses.

  proof: the expected number of satisfied clauses is $\sum_{i=1}^{m}(1 - 2^{-k_i})$

- There exists an n-vertex tournment[2] that has at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

  proof: let X be the number of Hamiltonian paths while X_i being the i^{th} HAM-path for $i = 1, 2, \ldots, n!$; then $E[X] = \frac{n!}{2^{n-1}}$

- Every n-vertex graph with minimum degree $d > 1$ has a dominating set of size at most $n\frac{1+\ln(d+1)}{d+1}$.

  proof: form a random vertex subset S by choosing each vertex from G with prob p; let T be the nodes that are not covered due to S; then, $E[|S \cup T|] \leq np + n(1-p)^{d+1} \leq np + ne^{-p(d+1)}$; set $p := \frac{\ln(d+1)}{d+1}$

- Crossing lemma: Let G be a simple graph with n vertices and m edges, where $m \geq 4n$. Then the crossing number of G is $\geq \frac{1}{64}\frac{m^3}{n^2}$

  proof: let n_p, m_p, c_p be the number of vertices, edges, and the crossings of a random subgraph generated by including vertices of G with prob p, independently from each other; then $E[c_p - m_p + 3n_p] = p^4 cr(G) - p^2 m + 3pn \geq 0$; set $p = \frac{4n}{m}$

- *Sample-and-Modify*: In the first stage, construct a random structure that does not have the required properties; in the second stage, modify the random structure so that it does have the required property.

- Let $G(V, E)$ be a simple graph. Then G has an independent set with at least $\frac{|V|}{2d}(= \frac{|V|^2}{4|E|})$ vertices, where d is the average degree of G.

  proof:

  (i) delete each vertex independently with prob $1 - \frac{1}{d}$; (ii) for each remaining edge, remove it and one of its adjacent vertices

[1] using the union bound

[2] giving direction to each edge (orienting each edge) in an undirected complete graph

let X and Y be the number of vertices and edges respectively that survive after the first stage; at the end of two stages, expected number of nodes left (an independent set size) is $E[X - Y] = \frac{|V|}{d} - \frac{|V|}{2d}$

- Assuming that n is sufficiently large, for any integer $k \geq 3$ there is a graph with n nodes, at least $\frac{1}{4}n^{1+\frac{1}{k}}$ edges, and girth at least k.

proof:

(i) construct a random graph in $G_{n,p}$ model with $p = n^{\frac{1}{k}-1}$; (ii) remove one edge from each cycle of length up to $k - 1$

let X and Y be the number of edges and cycles of length at most $k - 1$ respectively after the first stage; then $E[X - Y] \geq p\binom{n}{2} - \sum_{i=3}^{k-1} \binom{n}{i} \frac{(i-1)!}{2} p^i$

References:

Probability and Computing by M. Mitzenmacher and E. Upfal.