Introduction to Optimization

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Are you using optimization?

The word “optimization” may be very familiar or may be quite new to you.

……. but whether you know about optimization or not, you are using optimization in many occasions of your day to day life …….

…………….Examples………………

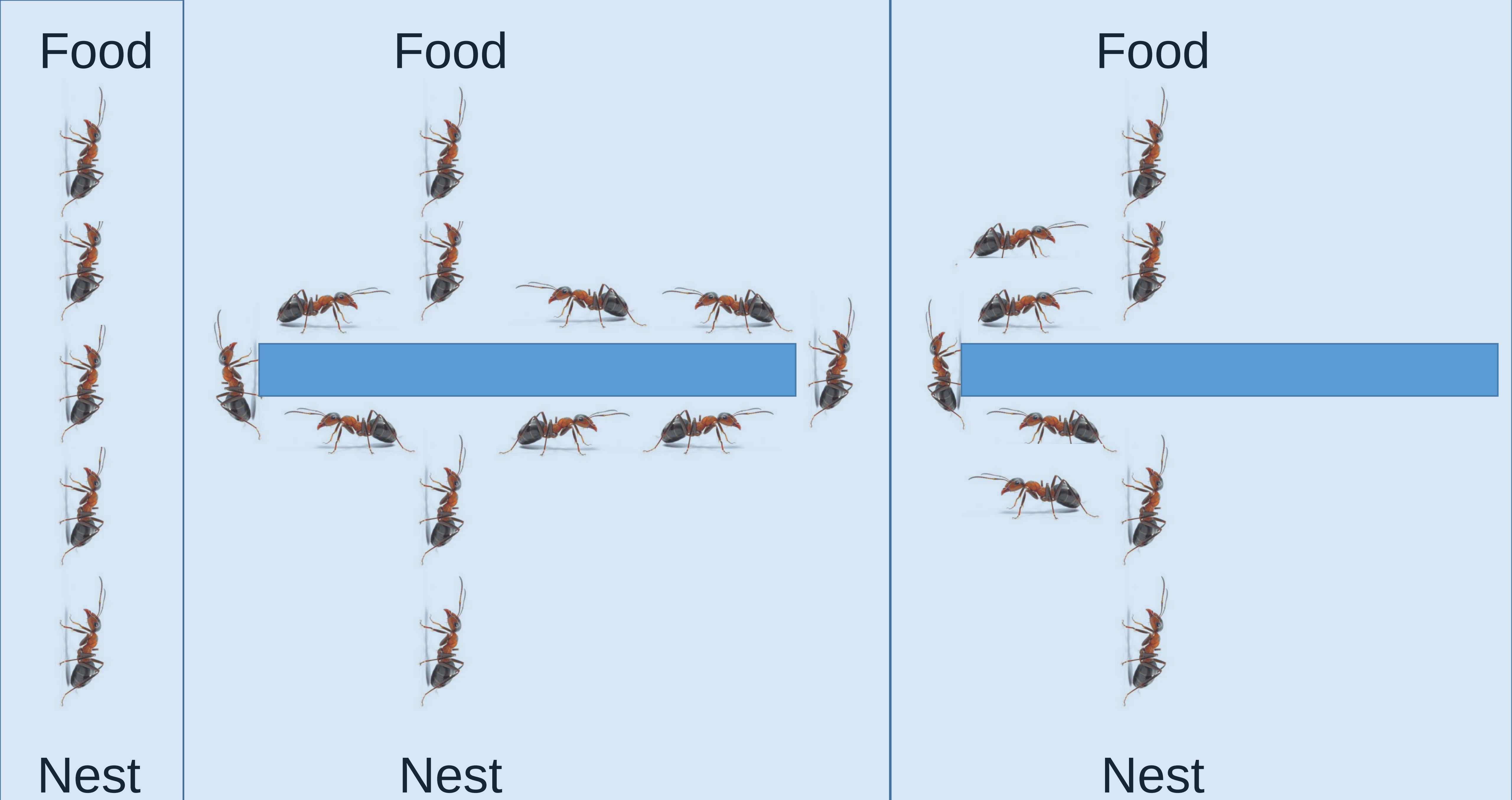
Optimization in real life

Newspaper hawker

Cooking

Forensic artist

Ant colony

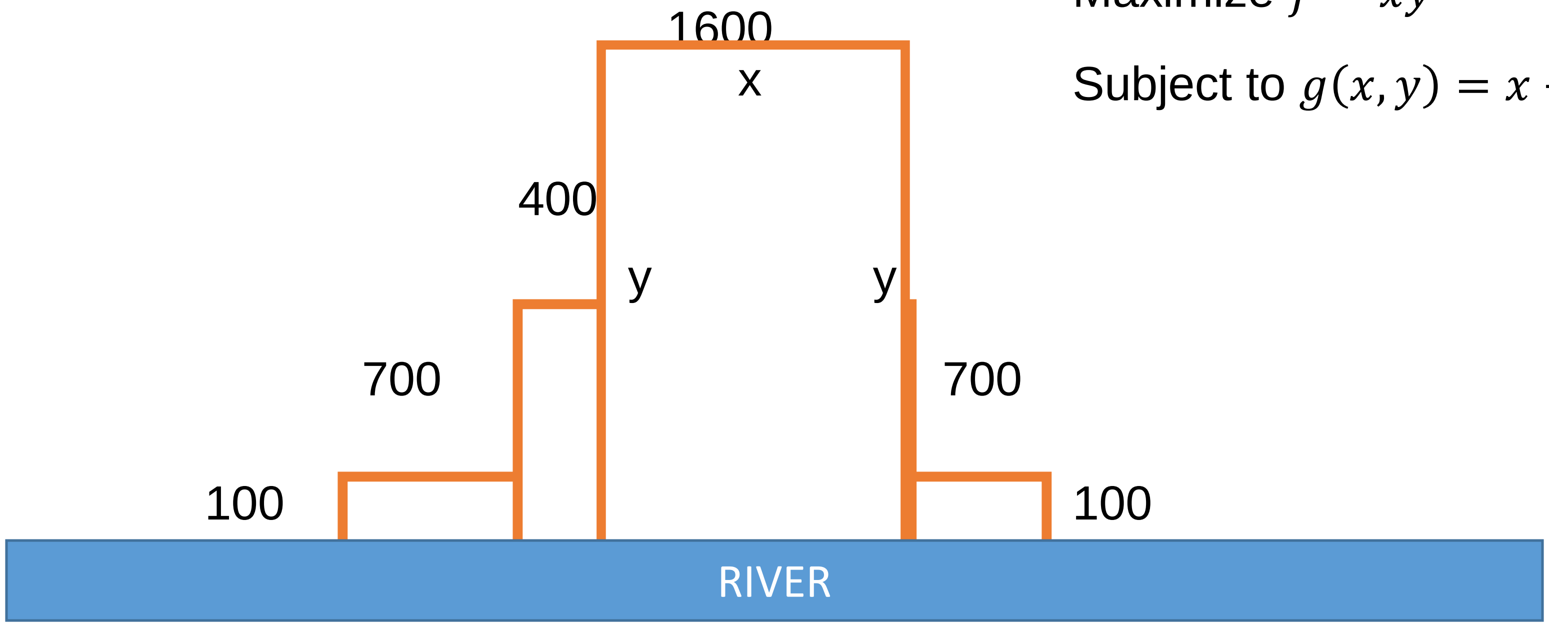

Food
Food
Food
Nest
Nest
Nest

Example

A farmer has 2400 m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

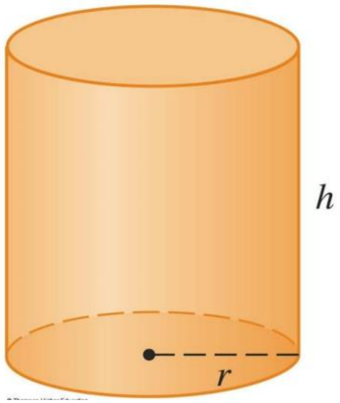
Maximize $f = xy$

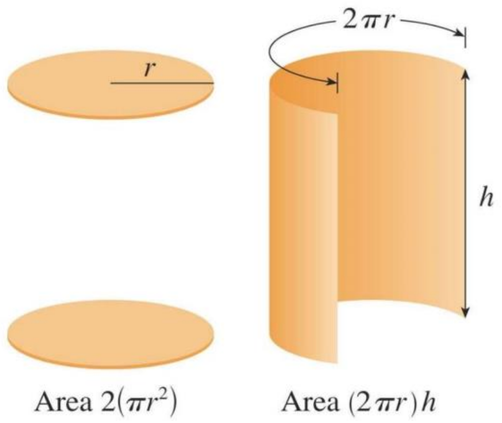
Subject to $g(x, y) = x + 2y = 2400$



Example

A manufacturer needs to make a cylindrical can that will hold 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.





Minimize: $A = 2\pi r^2 + 2\pi rh$

Constraint: $\pi r^2 h = 1500$

1.5 liters = 1500 cm^3

Dimension is in cm

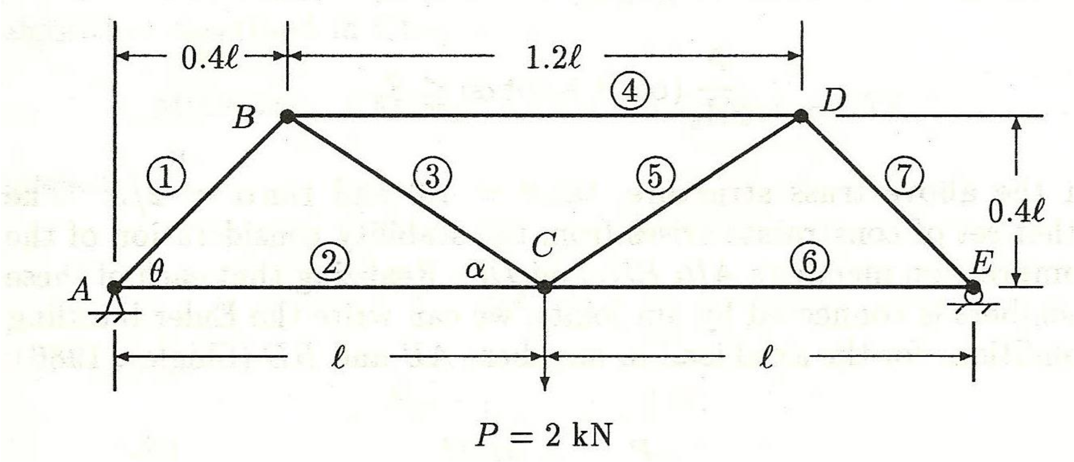
Example

Objectives

Topology: Optimal connectivity of the structure

Minimum cost of material: optimal cross section of all the members

We will consider the second objective only

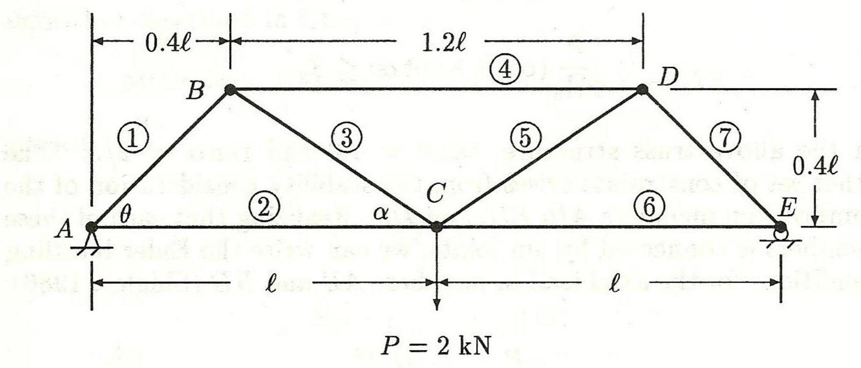


The design variables are the cross sectional area of the members, i.e. A_1 to A_7

Using symmetry of the structure $A_7 = A_1, A_6 = A_2, A_5 = A_3$

You have only four design variables, i.e., A_1 to A_4

Optimization formulation



Objective

$$Minimize\ f = 1.132A_1l + 2A_2l + 1.789A_3l + 1.2A_4l$$

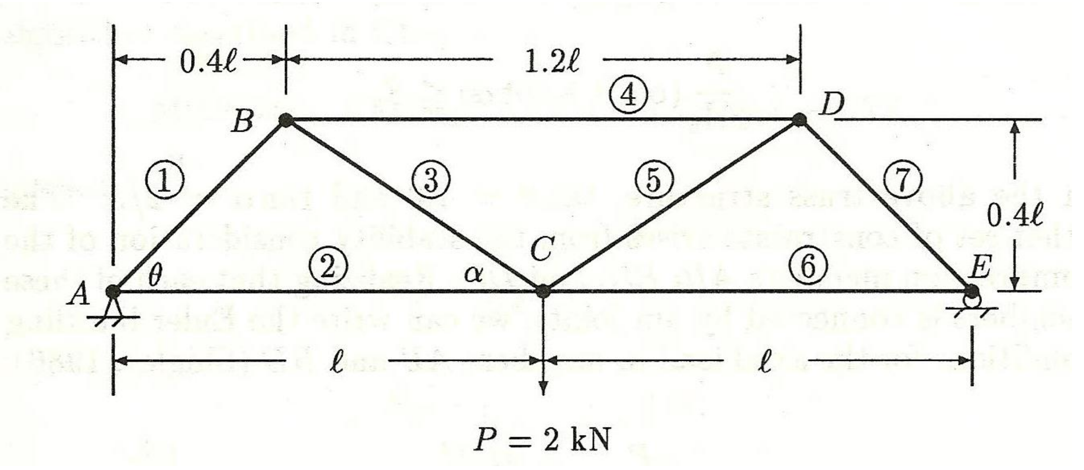
What are the constraints?

One essential constraint is non-negativity of design variables, i.e.
$A_1, A_2, A_3, A_4 \geq 0$

Is it complete now?

Optimization formulation

Member	Force	Member	Force
AB	$-\frac{P}{2}csc\theta$	BC	$+\frac{P}{2}csc\alpha$
AC	$+\frac{P}{2}cot\theta$	BD	$-\frac{P}{2}(cot\theta + cot\alpha)$



First set of constraints

$$\frac{Pcsc\theta}{2A_1} \le S_{yc}$$

$$\frac{Pcot\theta}{2A_2} \le S_{yt}$$

$$\frac{Pcsc\alpha}{2A_3} \le S_{yt}$$

$$\frac{P}{2A_4}(cot\theta + cot\alpha) \le S_{yc}$$

Another constraint is buckling of compression members

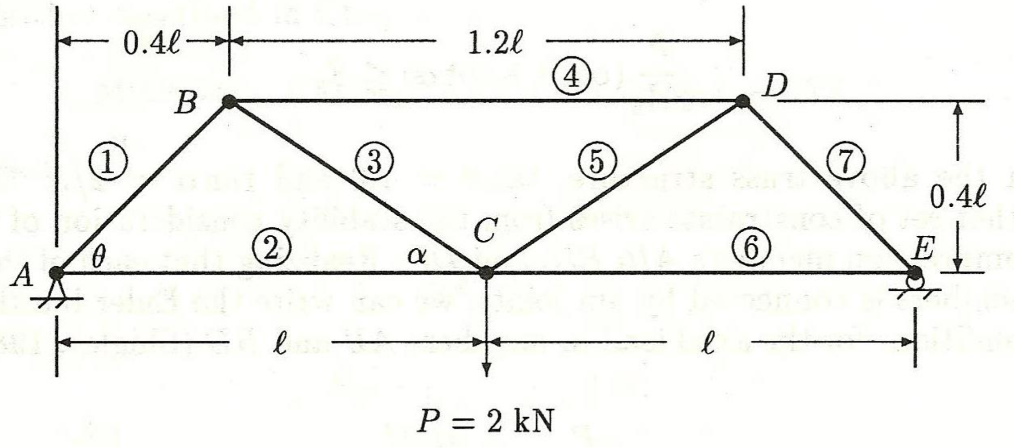
$$\frac{P}{2sin\theta} \le \frac{\pi E A_1^2}{1.281 l^2}$$

$$\frac{P}{2}(cot\theta + cot\alpha) \le \frac{\pi E A_4^2}{5.76 l^2}$$

Another constraint may be the minimization of deflection at C

$$\frac{Pl}{E}\left(\frac{0.566}{A_1} + \frac{0.500}{A_2} + \frac{2.236}{A_3} + \frac{2.700}{A_4}\right) \le \delta_{max}$$

Optimization formulation



$$Minimize\ f = 1.132A_1l + 2A_2l + 1.789A_3l + 1.2A_4l$$

Subject to

$$S_{yc} - \frac{Pcsc\theta}{2A_1} \geq 0$$

$$S_{yt} - \frac{Pcot\theta}{2A_2} \geq 0$$

$$S_{yt} - \frac{Pcsc\alpha}{2A_3} \geq 0$$

$$S_{yc} - \frac{P}{2A_4}(cot\theta + cot\alpha) \geq 0$$

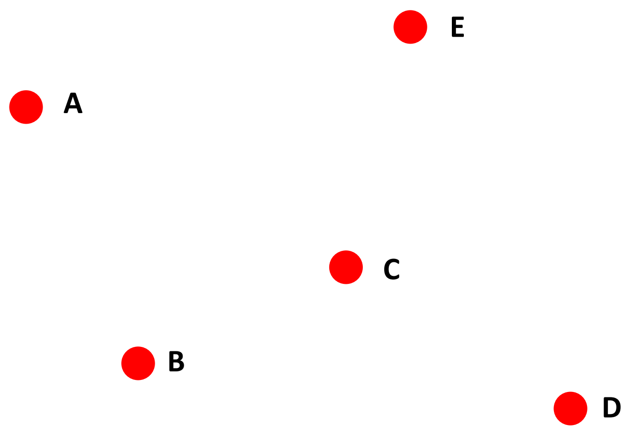
$$\frac{\pi E A_1^2}{1.281l^2} - \frac{P}{2sin\theta} \geq 0$$

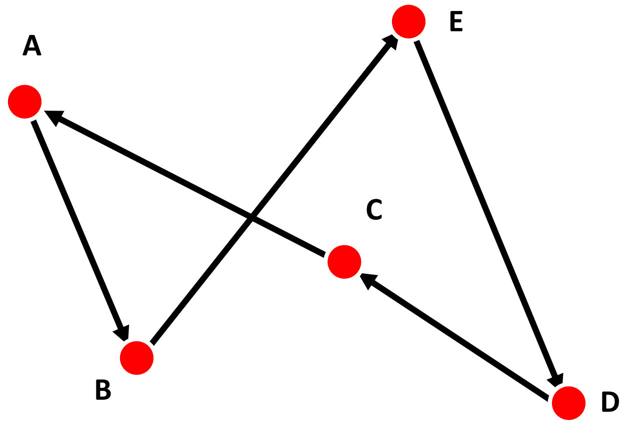
$$\frac{\pi E A_4^2}{5.76l^2} - \frac{P}{2}(cot\theta + cot\alpha) \geq 0$$

$$\delta_{max} - \frac{Pl}{E}\left(\frac{0.566}{A_1} + \frac{0.500}{A_2} + \frac{2.236}{A_3} + \frac{2.700}{A_4}\right) \geq 0$$

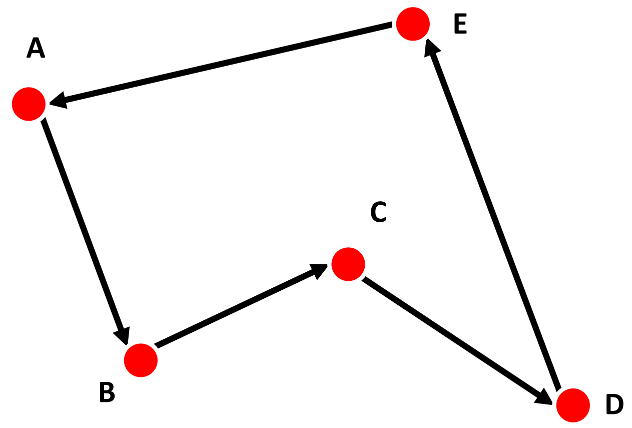
$$10 \times 10^{-6} \leq A_1, A_2, A_3,\ A_4 \leq 10 \times 10^{+6}$$

Traveling salesman problem





A possible solution: A B E D C



Another solution: A B C D E

Traveling salesman problem

Traveling salesman problem

What is Optimization?

- Optimization is the act of obtaining the best result under a given circumstances.

- Optimization is the mathematical discipline which is concerned with finding the maxima and minima of functions, possibly subject to constraints.

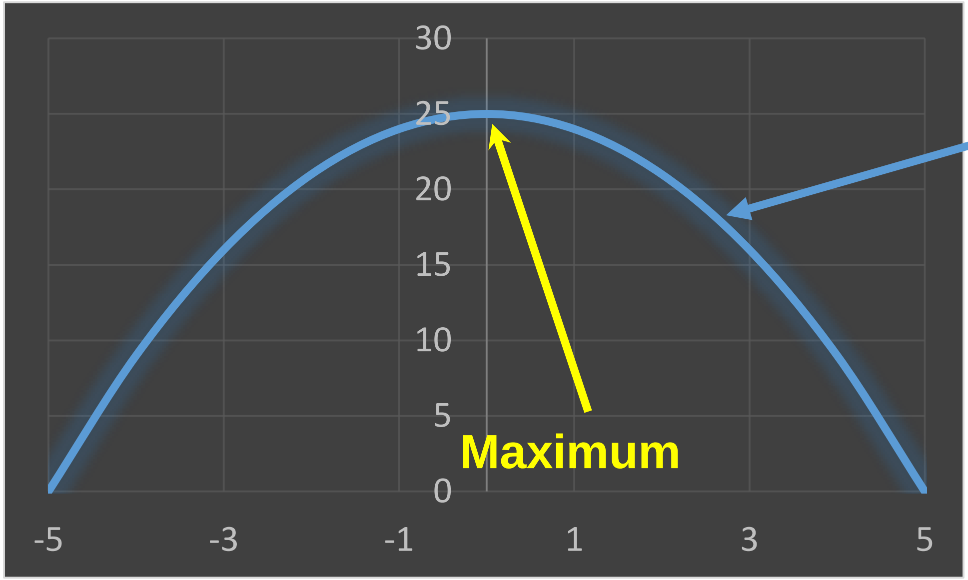
Introduction to optimization

$f = (x-5)^2$ Equation of the line

How to find out the minimum of the function

$f' = 2 \times (x-5) = 0$

$x^* = 5$ Optimal solution

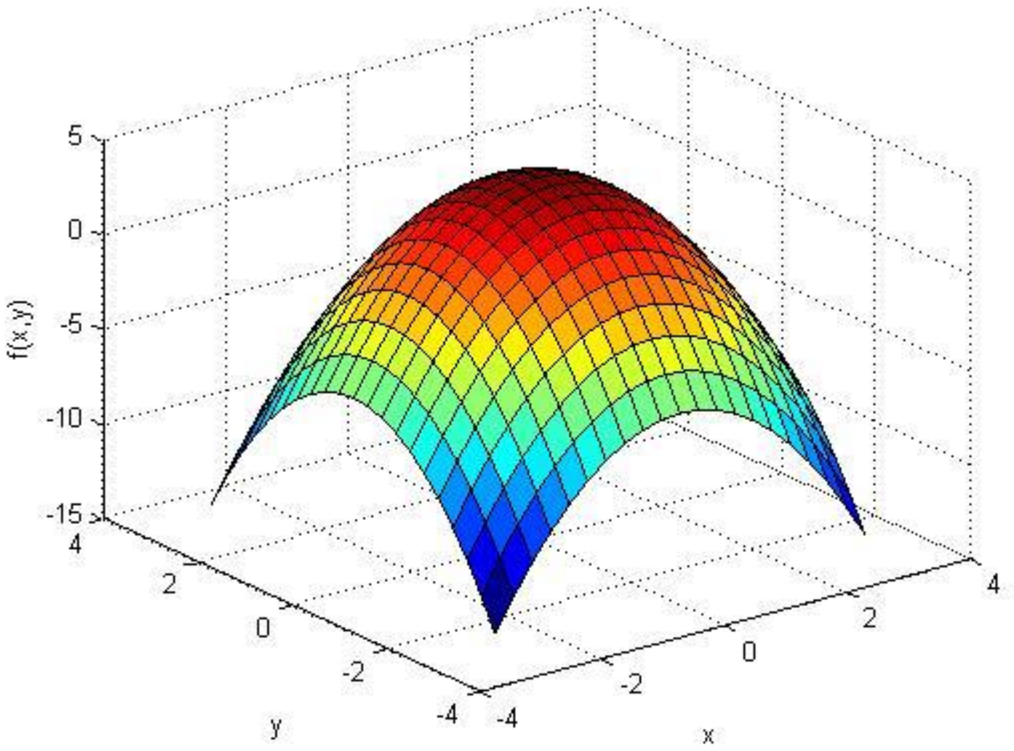


$f = 25 - x^2$ Equation of the line

$f' = 2x = 0$

$x^* = 0$ Optimal solution

Introduction to optimization



Equation of the surface

$$f(x, y) = -(x^2 + y^2) + 4$$

In this case, we can obtain the optimal solution by taking derivatives with respect to variable x and y and equating them to zero

$$\frac{\partial f}{\partial x} = -2x = 0 \qquad \Rightarrow x^* = 0$$

$$\frac{\partial f}{\partial y} = -2y = 0 \qquad \Rightarrow y^* = 0$$

Optimal solution is $(0,0)$

Single variable optimization

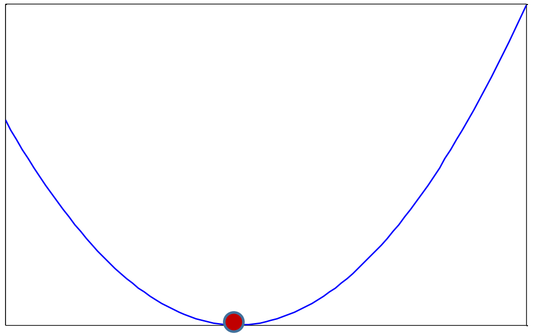
Objective function is defined as

Minimization/Maximization $f(x)$

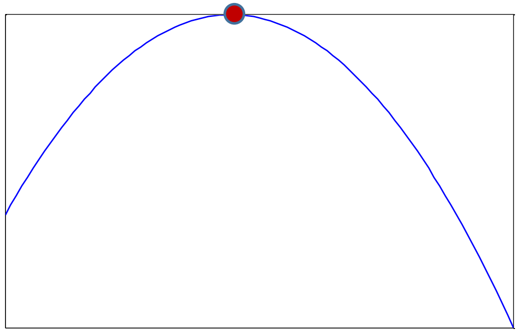
Single variable optimization

Stationary points

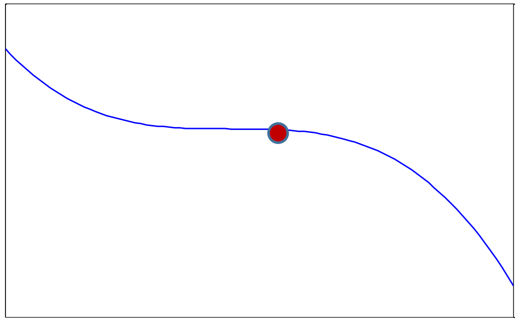
For a continuous and differentiable function $f(x)$, *a stationary point* x^* *is a point at which the* slope of the function is zero, *i.e.* $f'(x) = 0$ *at* $x = x^*$,



Minima



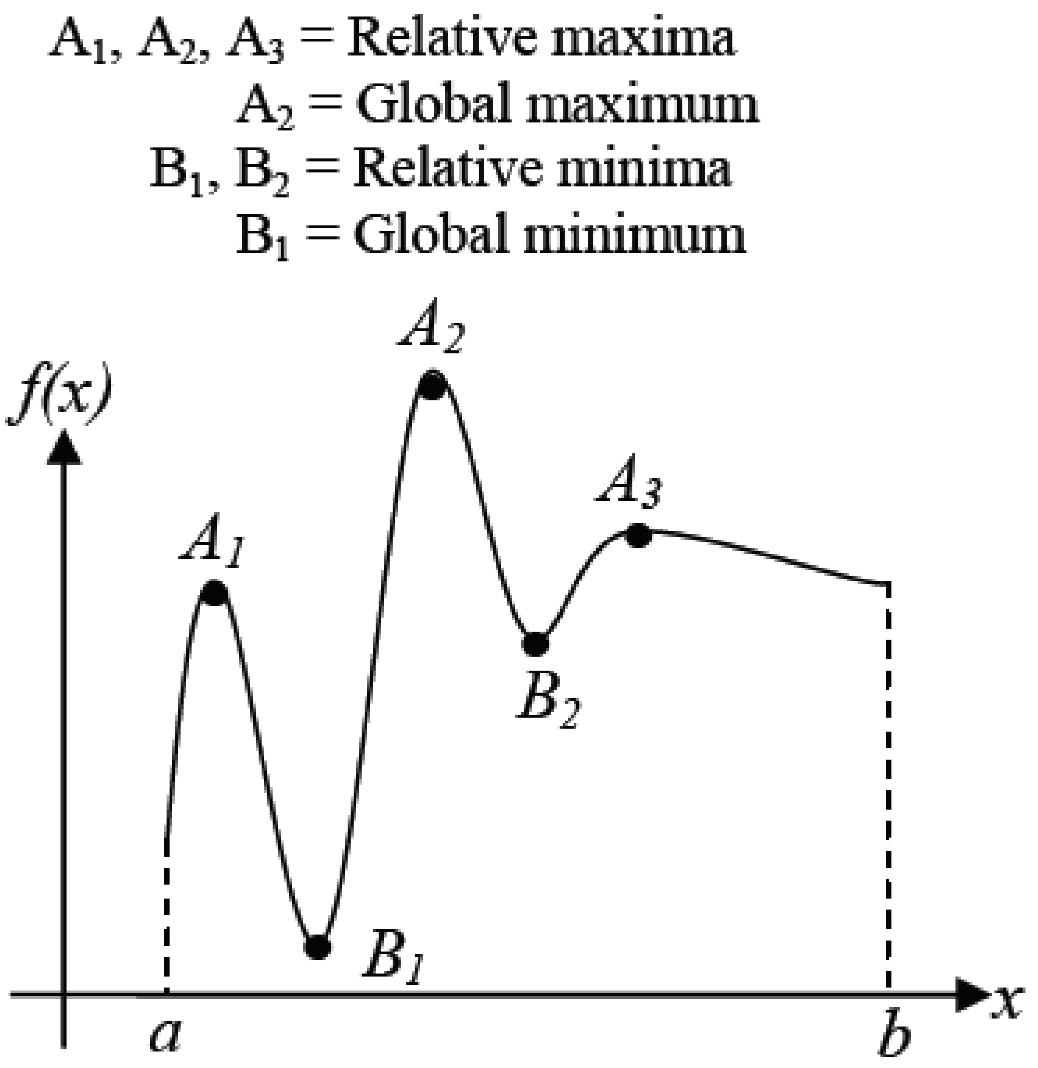
Maxima



Inflection point

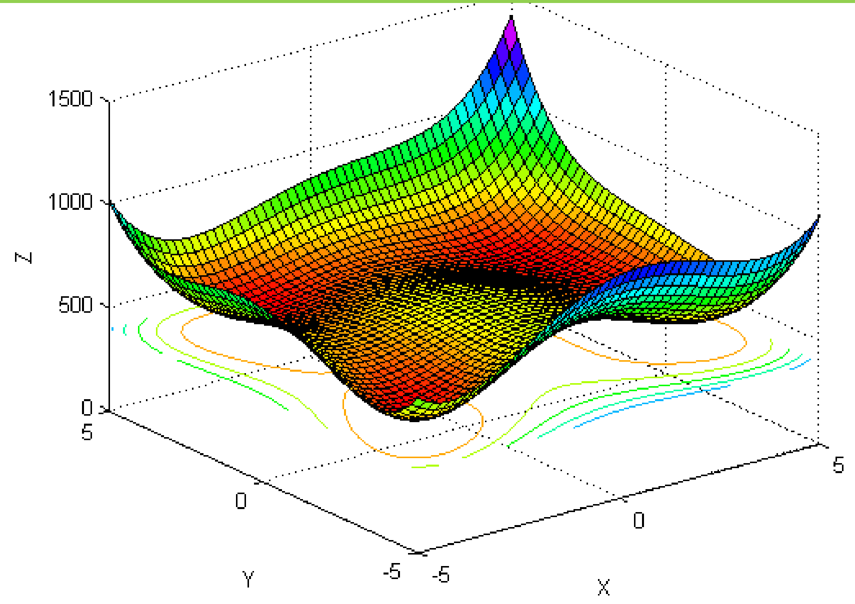
Global minimum and maximum

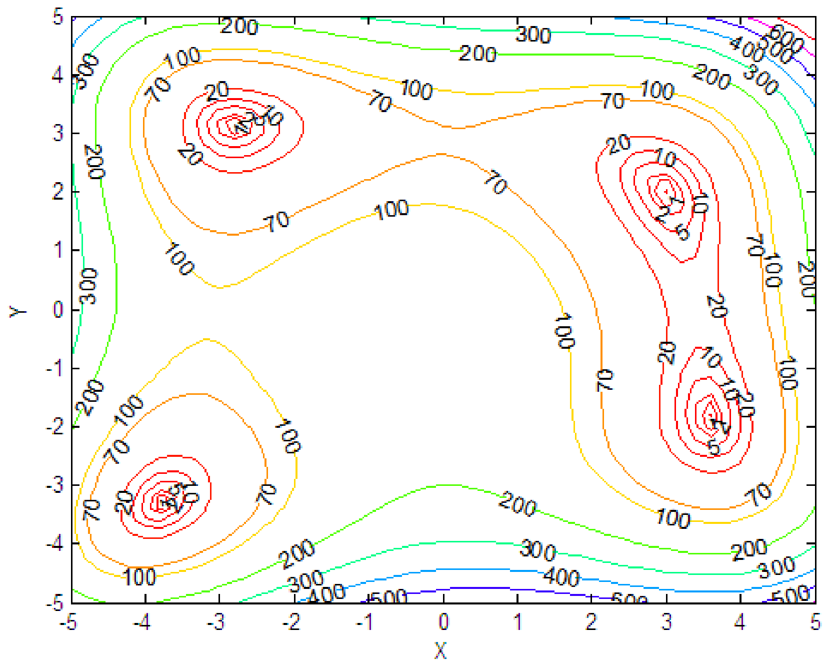
A function is said to have a *global or absolute minimum at* $x = x^*$ *if* $f(x^*) \leq f(x)$ *for all* x *in the domain over which* $f(x)$ *is defined.*

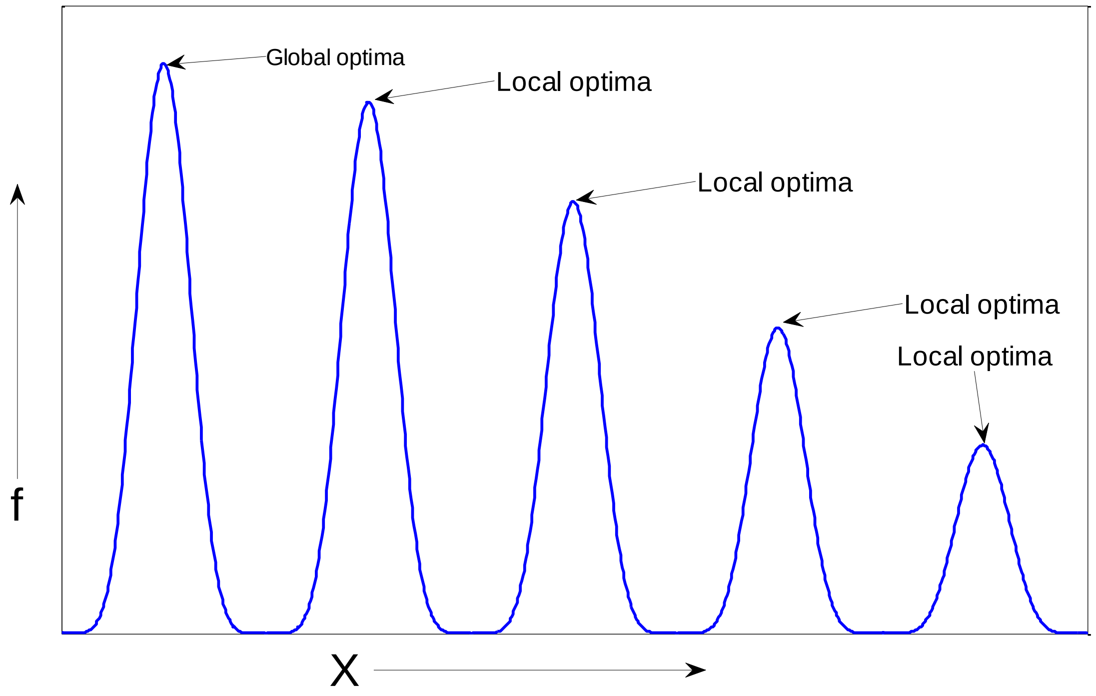


A function is said to have a *global or absolute maximum at* $x = x^*$ *if* $f(x^*) \geq f(x)$ *for all* x *in the domain over which* $f(x)$ *is defined.*

Introduction to optimization







Necessary condition

If a function $f(x)$ is defined in the interval $a \leq x \leq b$ and has a relative minimum at $x = x^*$, Where $a \leq x^* \leq b$ and if $f^/(x)$ exists as a finite number at $x = x^*$, then $f^/(x^*) = 0$

Proof

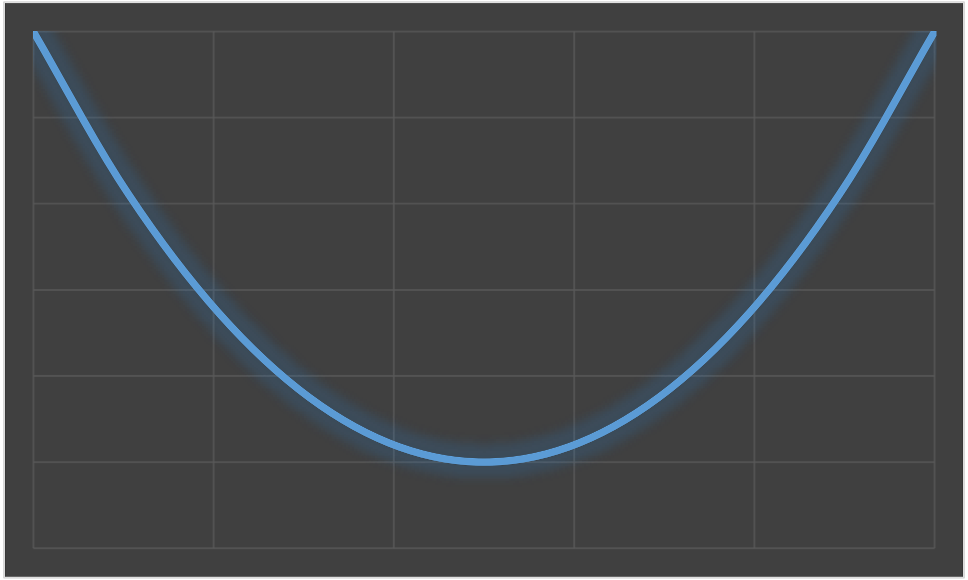
$$f^/(x^*) = \lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h}$$

Since x^* is a relative minimum $\quad f(x^*) \leq f(x^* + h)$

For all values of h sufficiently close to zero, hence

$$\frac{f(x^* + h) - f(x^*)}{h} \geq 0 \qquad \text{if } h \geq 0$$

$$\frac{f(x^* + h) - f(x^*)}{h} \leq 0 \qquad \text{if } h \leq 0$$



Necessary and sufficient conditions for optimality

Thus

$f^{/}(x^*) \geq 0$ If h tends to zero through +ve value

$f^{/}(x^*) \leq 0$ If h tends to zero through -ve value

Thus only way to satisfy both the conditions is to have

$$\boldsymbol{f^{/}(x^*) = 0}$$

Note:

- This theorem can be proved if x^* is a relative maximum
- Derivative must exist at x^*
- The theorem does not say what happens if a minimum or maximum occurs at an end point of the interval of the function
- It may be an inflection point also.

Sufficient conditions for optimality

Sufficient condition

Suppose at point x^*, the first derivative is zero and first nonzero higher derivative is denoted by n, then

1. *If n is odd, x^* is an inflection point*
2. *If n is even, x^* is a local optimum*
   - ✓ *If the derivative is positive, x^* is a local minimum*
   - ✓ *If the derivative is negative, x^* is a local maximum*

$$f'(x^*) = 0$$

$$f''(x^*) = 0$$

$$f^3(x^*) = 0$$

$$f^4(x^*) = 0$$

$$f^n(x^*) \neq 0$$

Sufficient conditions for optimality

Proof

Apply Taylor's series

$$f(x^* + h) = f(x^*) + hf'(x^*) + \frac{h^2}{2!}f''(x^*) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(x^*) + \frac{h^n}{n!}f^n(x^*)$$

Since $f'(x^*) = f''(x^*) = \cdots = f^{n-1}(x^*) = 0$

$$f(x^* + h) - f(x^*) = \frac{h^n}{n!}f^n(x^*)$$

When n is even $\frac{h^n}{n!} \geq 0$

Thus if $f^n(x^*)$ is positive $f(x^* + h) - f(x^*)$ is positive Hence it is local minimum

Thus if $f^n(x^*)$ negative $f(x^* + h) - f(x^*)$ is negative Hence it is local maximum

When n is odd, $\left(\frac{h^n}{n!}\right)$ changes sign with the change in the sign of h.

Hence it is an inflection point

Sufficient conditions for optimality

Take an example

$$f(x) = x^3 - 10x - 2x^2 - 10$$

Apply necessary condition $f'(x) = 3x^2 - 10 - 4x = 0$

Solving for x $x^* = 2.61\ and\ -1.28$ These two points are stationary points

Apply sufficient condition $f''(x) = 6x - 4$

$f''(2.61) = 11.66$ *positive and n is even* $f''(-1.28) = -11.68$ *negative and n is ever*

$x^* = 2.61$ is a minimum point

$x^* = -1.28$ is a maximum point

Multivariable optimization without constraints

Minimize $f(X)$ Where $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Necessary condition for optimality

If $f(X)$ has an extreme point (maximum or minimum) at $X = X^*$ and if the first partial Derivatives of $f(X)$ exists at X^*, then

$$\frac{\partial f(X^*)}{\partial x_1} = \frac{\partial f(X^*)}{\partial x_2} = \cdots = \frac{\partial f(X^*)}{\partial x_n} = 0$$

Multivariable optimization without constraints

Sufficient condition for optimality

The sufficient condition for a stationary point X^* to be an extreme point is that the matrix of second partial derivatives of $f(X)$ evaluated at X^* is

(1) positive definite when X^* is a relative minimum
(2) negative definite when X^* is a relative maximum
(3) neither positive nor negative definite when X^* is neither a minimum nor a maximum

Proof Taylor series of two variable function

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y} + \frac{1}{2!}\left(\Delta x^2 \frac{\partial^2 f}{\partial x^2} + 2\Delta x \Delta y \frac{\partial^2 f}{\partial x \partial y} + \Delta y^2 \frac{\partial^2 f}{\partial y^2}\right) + \cdots$$

$$f(x+\Delta x, y+\Delta y) = f(x,y) + [\Delta x \quad \Delta y]\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} + \frac{1}{2!}[\Delta x \quad \Delta y]\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + \cdots$$

Multivariable optimization without constraints

$$f(X^*+h) = f(X^*) + h^T \nabla f(X^*) + \frac{1}{2!} h^T \boldsymbol{H} h + \cdots$$

Since X^* is a stationary point, the necessary condition gives that $\nabla f(X^*) = 0$

Thus

$$f(X^*+h) - f(X^*) = \frac{1}{2!} h^T \boldsymbol{H} h + \cdots$$

Now, X^* will be a minima, if $h^T \boldsymbol{H} h$ is positive

X^* will be a maxima, if $h^T \boldsymbol{H} h$ is negative

$h^T \boldsymbol{H} h$ will be positive if $\boldsymbol{H}$ is a positive definite matrix

$h^T H h$ will be negative if $\boldsymbol{H}$ is a negative definite matrix

A matrix H will be positive definite if all the eigenvalues are positive, *i.e.* all the λ values are positive which satisfies the following equation

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$$

Another test

$$A_1 = |a_{11}|$$

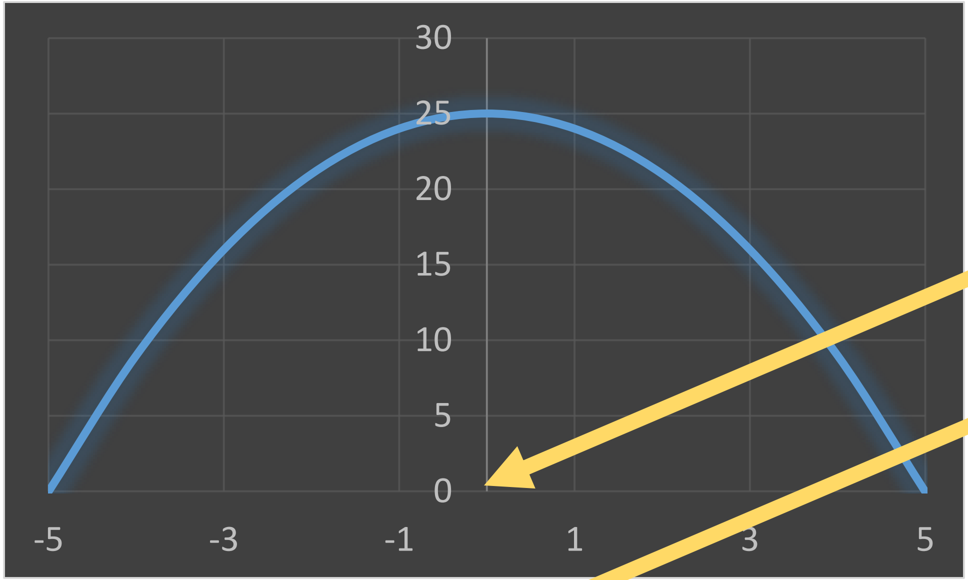
$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$A_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{vmatrix}$$

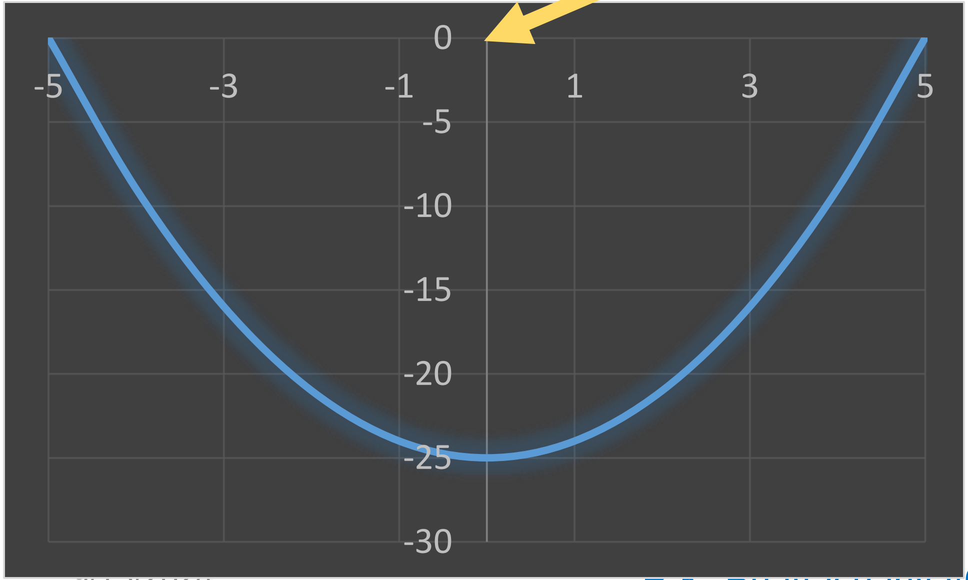
- ✓ A matrix $\boldsymbol{A}$ will be positive definite if any only if all the values $A_1, A_2, A_3, \ldots, A_n$ are positive.
- ✓ The matrix will be negative definite is and only if the sign of A_j is $(-1)^j$ for $j = 1, 2, 3, \ldots, n$

Unimodal and duality principle



Optimal solution $x^* = 0$

Optimal solution $x^* = 0$



Minimization $f(x)$ = Maximization $-f(x)$

Thank you