Linear Problem (LP)

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Linear Problem (LP)

Minimization $f(X)$

Maximization $f(x)$ =Minimization$-f(x)$

Subject to $g_j(X) \leq 0 \quad j = 1, 2, 3, \ldots, J$

$$h_k(X) = 0 \quad k = 1, 2, 3, \ldots, K$$

If $f(X), g(X)$ *and* $h(X)$ are linear, the problem is called a linear problem. Else the problem is a non-linear problem

$$g(X) \geq 0 \quad \Rightarrow -g(X) \leq 0$$

Linear Problem (LP)

Linear programming

It is an optimization method applicable for the solution of optimization problem where objective function and the constraints are linear

It was first applied in 1930 by economist, mainly in solving resource allocation problem

During World War II, the US Air force sought more effective procedure for allocation of resources

George B. Dantzig, a member of the US Air Force formulate general linear problem for solving the resources allocation problem.

The devised method is known as Simplex method

Linear programming

It is considered as a revolutionary development that helps in obtaining optimal decision in complex situation

Some of the great contributions are

George B. Dantzig : Devised simplex method
Kuhn and Tucker : Duality theory in LP
Charnes and Cooper: Industrial application of LP
Karmarkar : Karmarkar's method

Nobel prize awarded for contribution related to LP

Nobel prize in economics was awarded in 1975 jointly to L. V. Kantorovich of the former Soviet Union and T. C. Koopmans of USA on the application of LP to the economic problem of resource allocation.

Linear programming

Standard form of Linear Problem (LP)

Minimize $f(x_1, x_2, x_3, \dots, x_n) = c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n$

Subject to

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

$$x_1, x_2, x_3, \dots, x_n \geq 0$$

Linear programming

Standard form of Linear Problem (LP) in Matrix form

Minimize $f(X) = c^T X$

Subject to

$$aX = b$$

$$X \geq 0$$

Where

$$X = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \qquad b = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix} \qquad c = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{Bmatrix} \qquad a = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Linear programming

Standard form

1. The objective function is minimization type
2. All constraints are equality type
3. All the decision variables are non-negative

Linear programming

Standard form

1. The objective function is minimization type

For maximization problem

Maximize $f(x_1, x_2, x_3, \ldots, x_n) = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n$

Equivalent to

Minimize $F = -f(x_1, x_2, x_3, \ldots, x_n) = -c_1x_1 - c_2x_2 - c_3x_3 - \ldots - c_nx_n$

Linear programming

Standard form

2. All constraints are equality type

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + \cdots + a_{kn}x_n = b_k$$

If it is less than type

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + \cdots + a_{kn}x_n \leq b_k$$

It can be converted to

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + \cdots + a_{kn}x_n + x_{n+1} = b_k$$

Slack variable

If it is greater than type

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + \cdots + a_{kn}x_n \geq b_k$$

It can be converted to

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + \cdots + a_{kn}x_n - x_{n+1} = b_k$$

Surplus variable

Linear programming

Standard form

3. All the decision variables are non-negative

$$x_1, x_2, x_3, \dots, x_n \geq 0$$

Is any variable x_j is unrestricted in sign, it can be expressed as

$$x_j = x_j' - x_j''$$

Where, $x_j', x_j'' \geq 0$

Linear programming

There are m equations and n decision variable

Now see the conditions

- ✓ If $m > n$, there will be $m - n$ redundant equations which can be eliminated
- ✓ If $m = n$, there will be an unique solution or there may not be any solution
- ✓ If $m < n$, a case of undetermined set of linear equations, if they have any solution, there may be innumerable solutions

The problem of linear programming is to find out the best solution that satisfy all the constraints

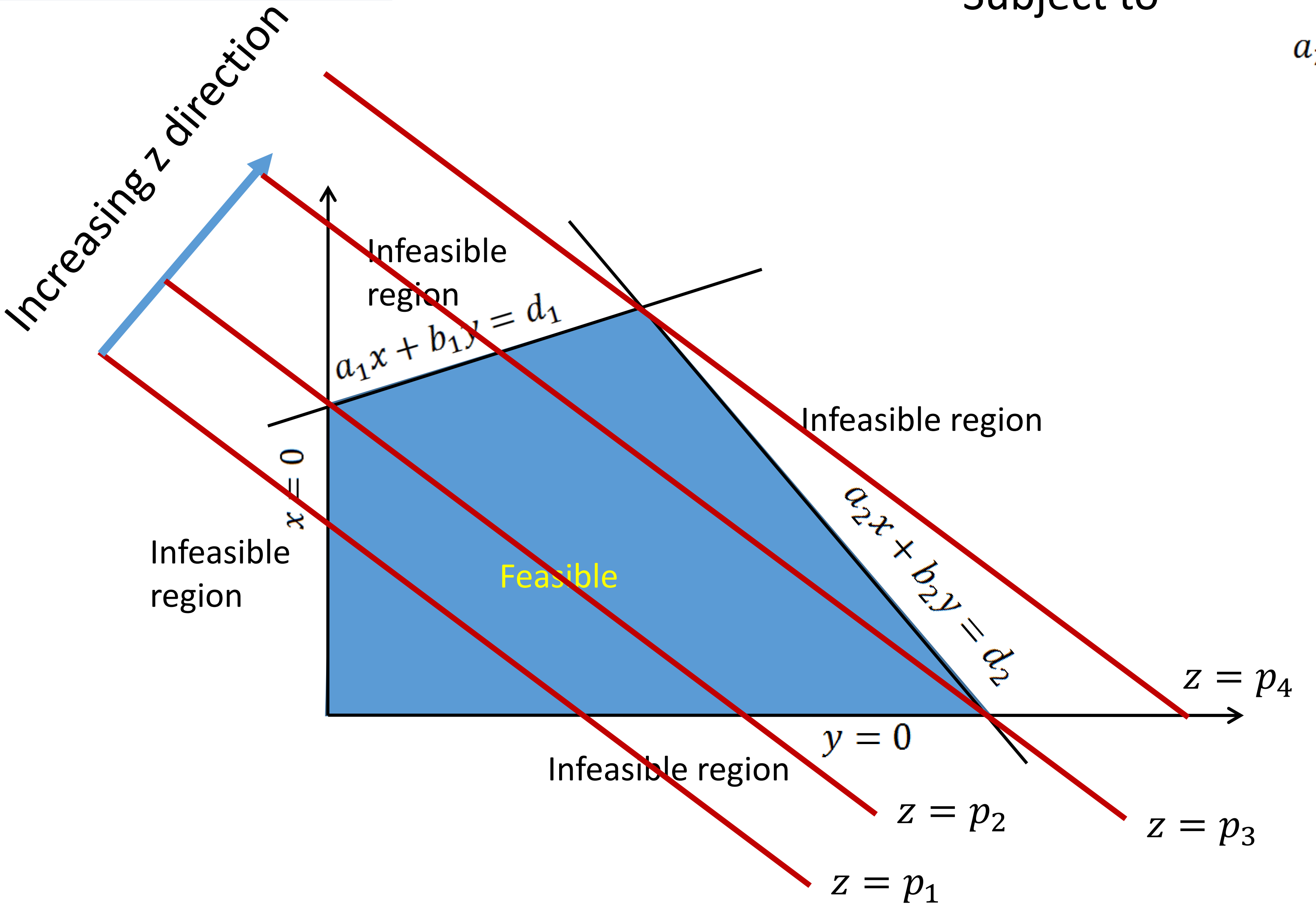
Linear Problem (LP)

Maximize $z = c_1 x + c_2 y$

Subject to $a_1 x + b_1 y \leq d_1$

$a_2 x + b_2 y \leq d_2$

$x, y \geq 0$



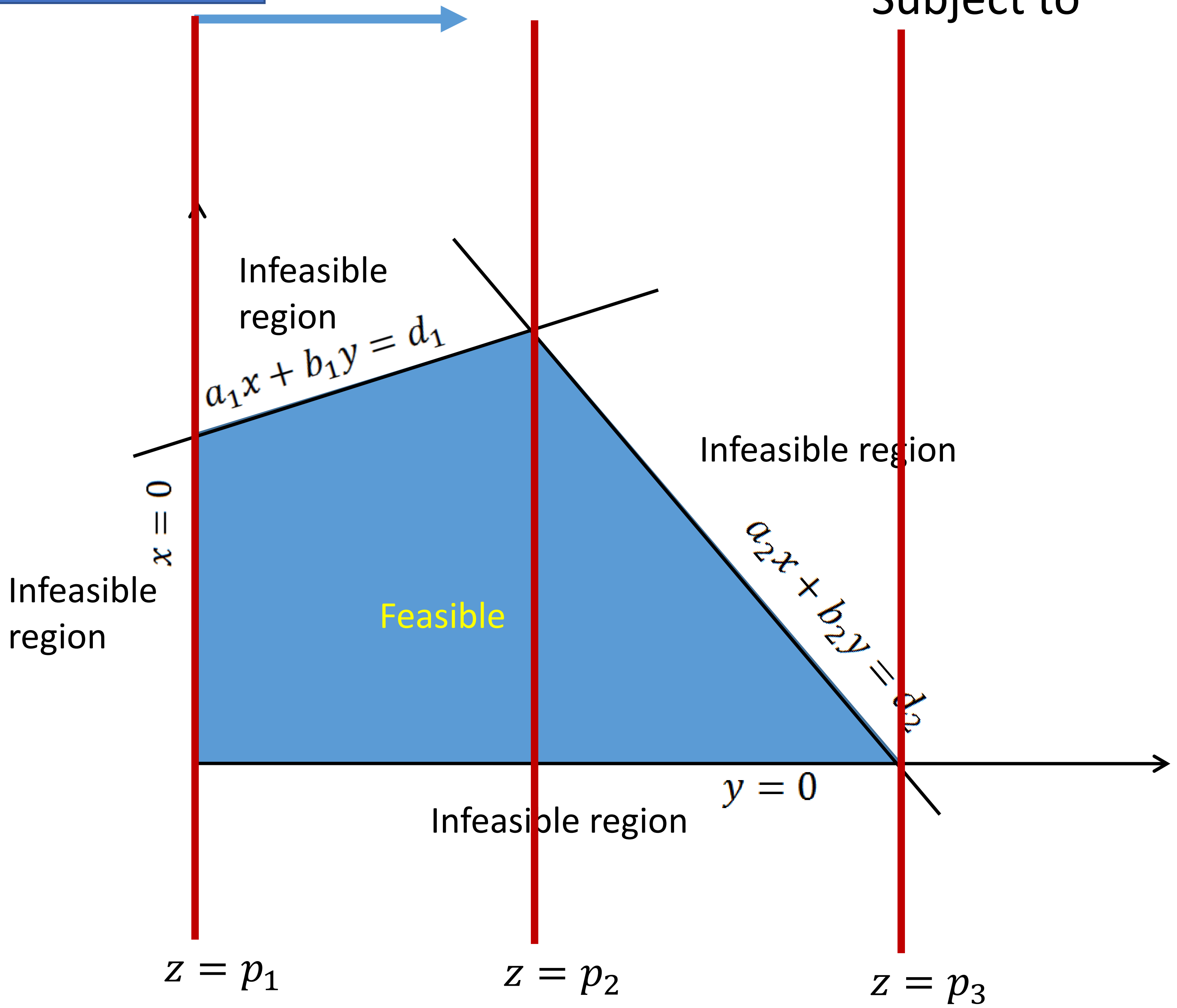
Linear Problem (LP)

Maximize $z = c_1 x + c_2 y$

Subject to

$$a_1 x + b_1 y \le d_1$$
$$a_2 x + b_2 y \le d_2$$
$$x, y \ge 0$$

Increasing z direction



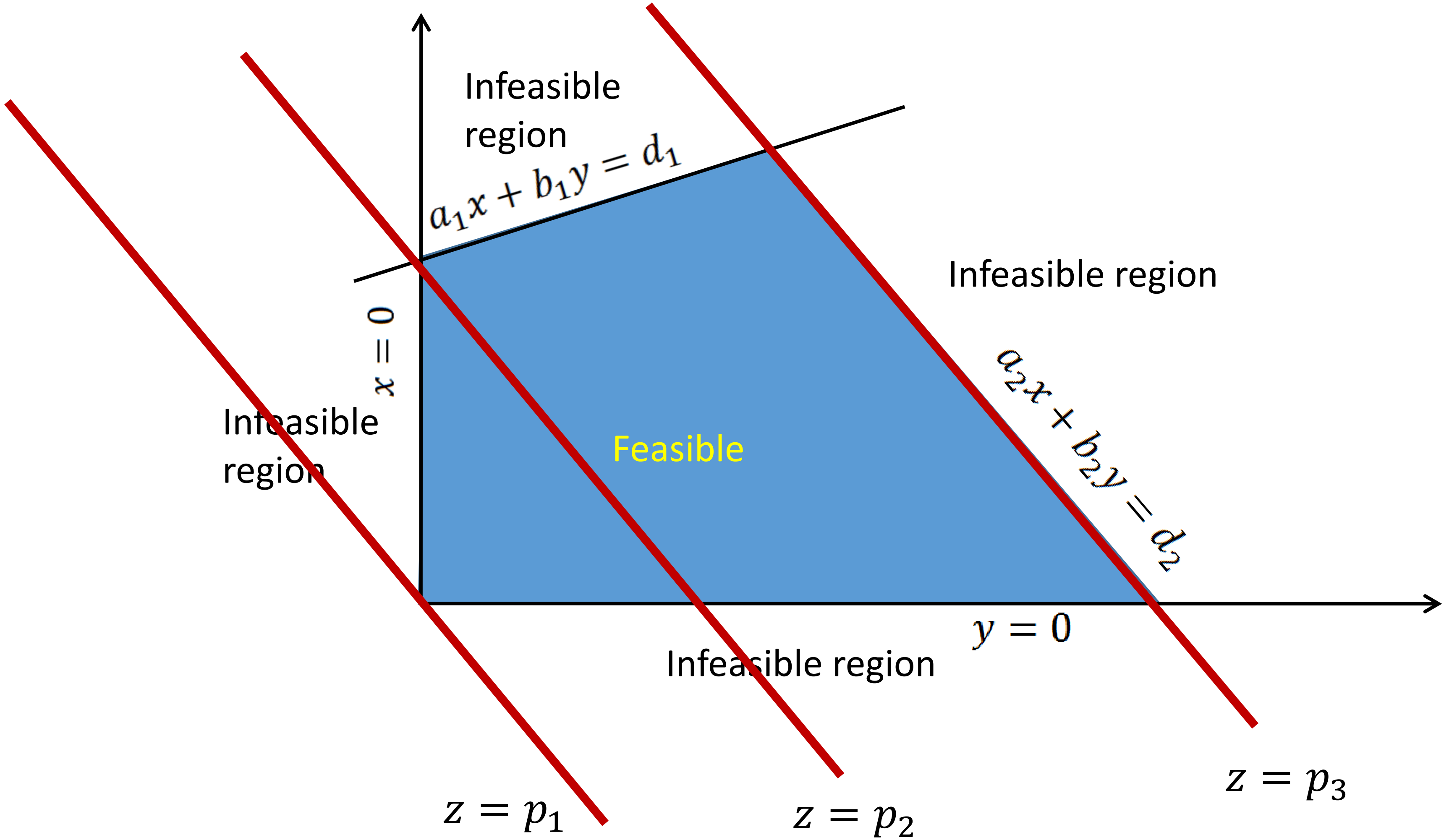
As such solution of the problem will be one of the corners of the search space

Linear Problem (LP)

Maximize $z = c_1x + c_2y$

Subject to

$$a_1x + b_1y \le d_1$$
$$a_2x + b_2y \le d_2$$
$$x, y \ge 0$$



This problem has infinite number of solutions

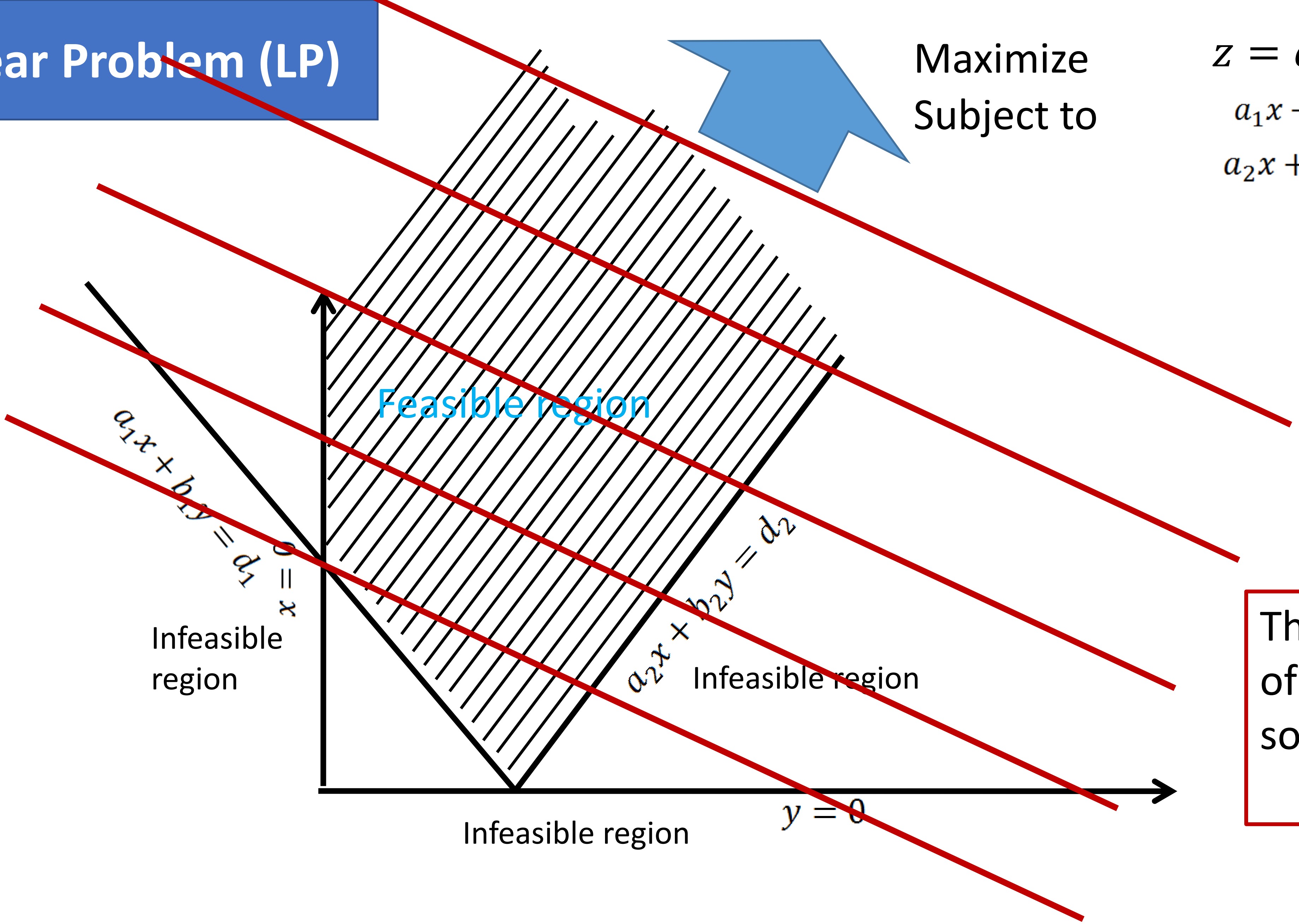
Linear Problem (LP)

Maximize $z = c_1x + c_2y$

Subject to $a_1x + b_1y \leq d_1$

$a_2x + b_2y \leq d_2$

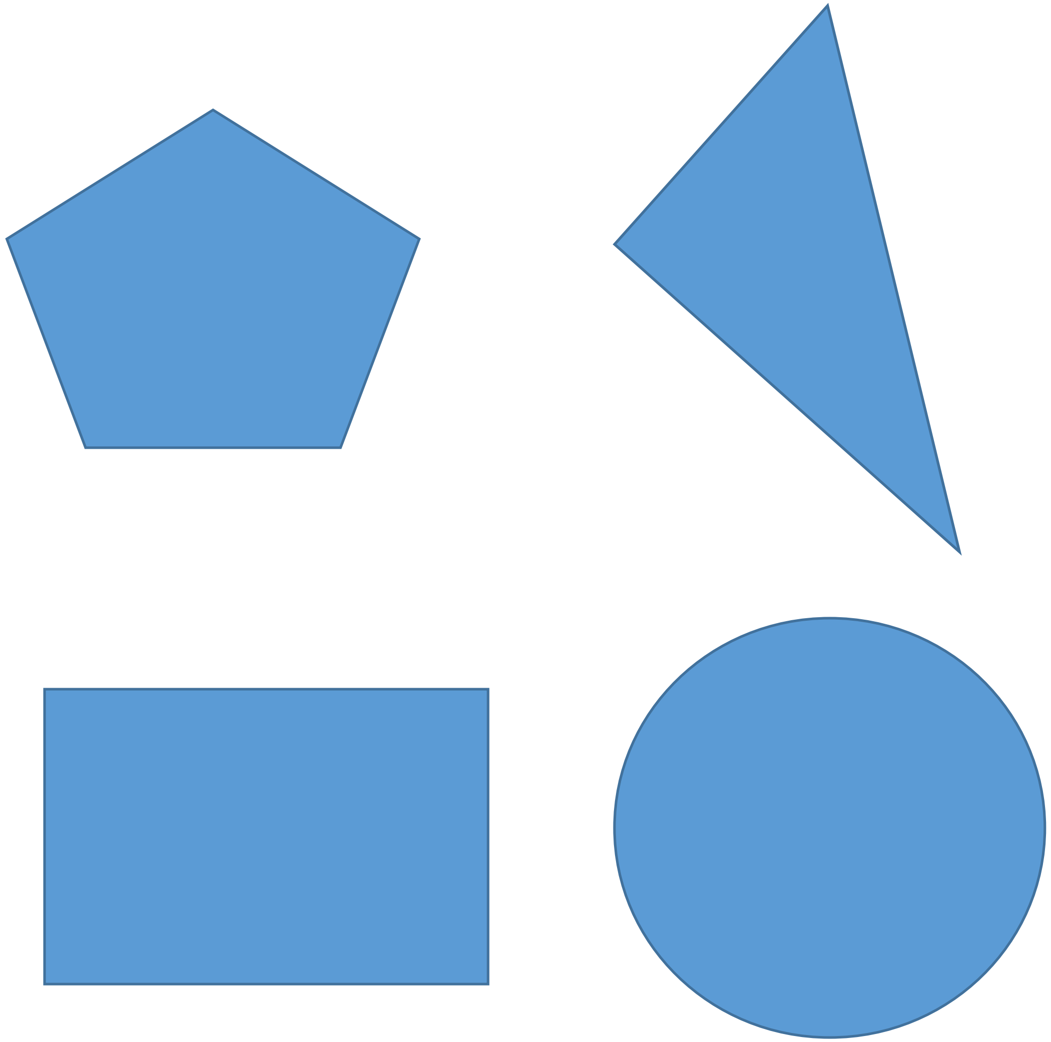
$x, y \geq 0$



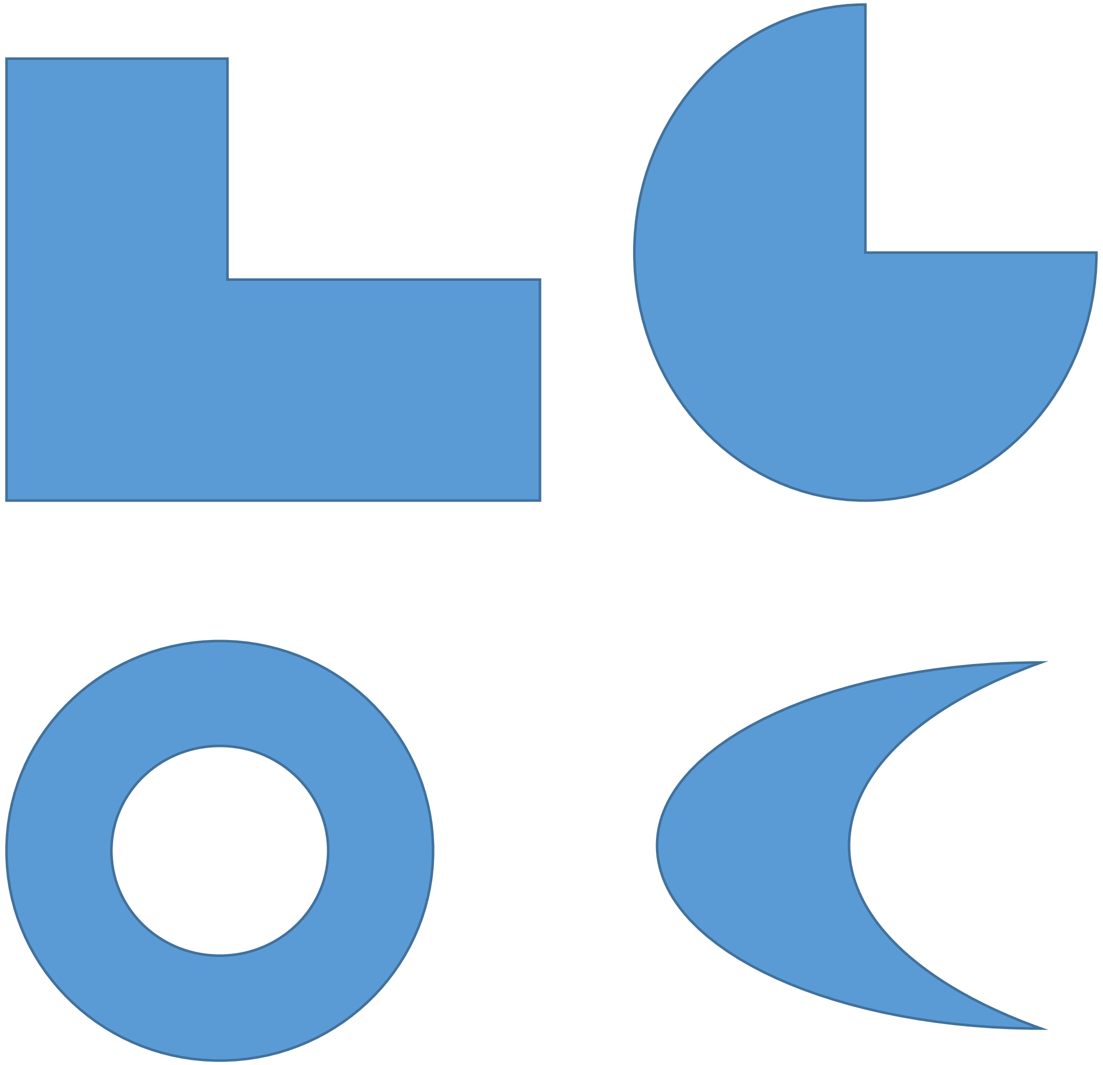
This is a problem of unbounded solution

Linear Problem (LP)

Search space



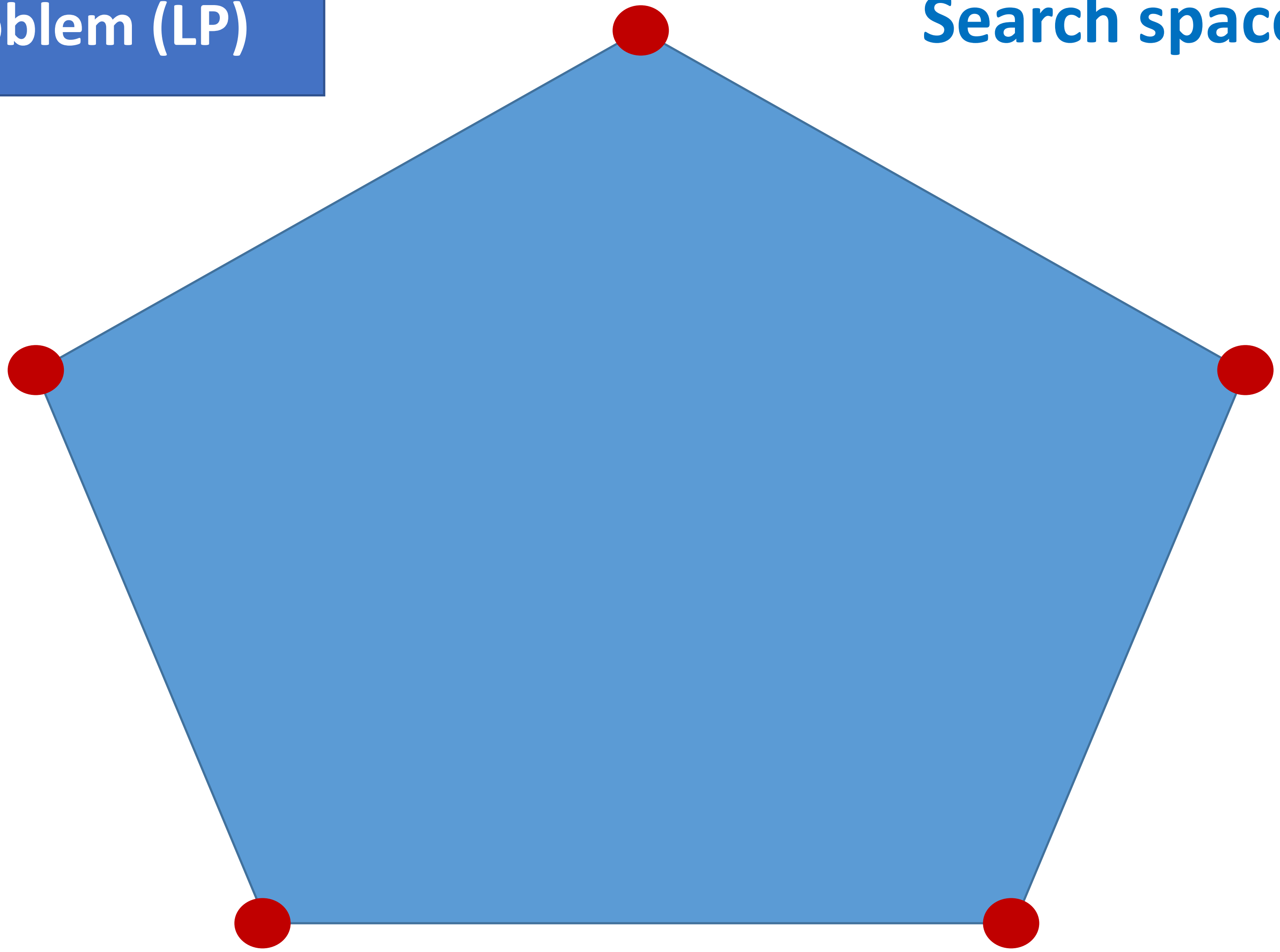
Convex Search space



Non Convex Search space

Linear Problem (LP)

Search space



Some definitions

Point of n -Dimensional space

A point X in an n -dimensional space is characterized by an ordered set of n values or coordinates. The coordinate of X are also called the component of X.

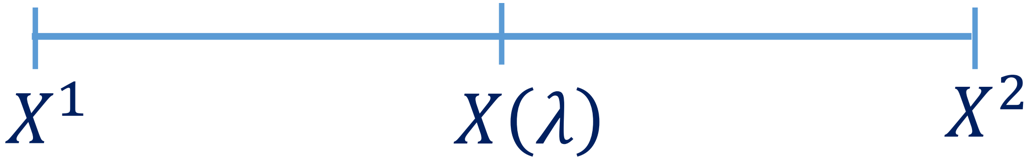
Line segment in n-Dimensions (L)

If coordinates of two pints X^1 and X^2 are given, the line segment (L) joining these points is the collection of points $X(\lambda)$ whose coordinates are given by

$$X(\lambda) = \lambda X^1 + (1 - \lambda) X^2$$

Thus $\mathrm{L} = \{X | X = \lambda X^1 + (1 - \lambda) X^2\}$

$$0 \leq \lambda \leq 1$$



Some definitions

Hyperplane

In n -dimensional space, the set of points whose coordinate satisfy a linear equation

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = a^TX = b$$

is called a hyperplane

A hyperplane is represented by

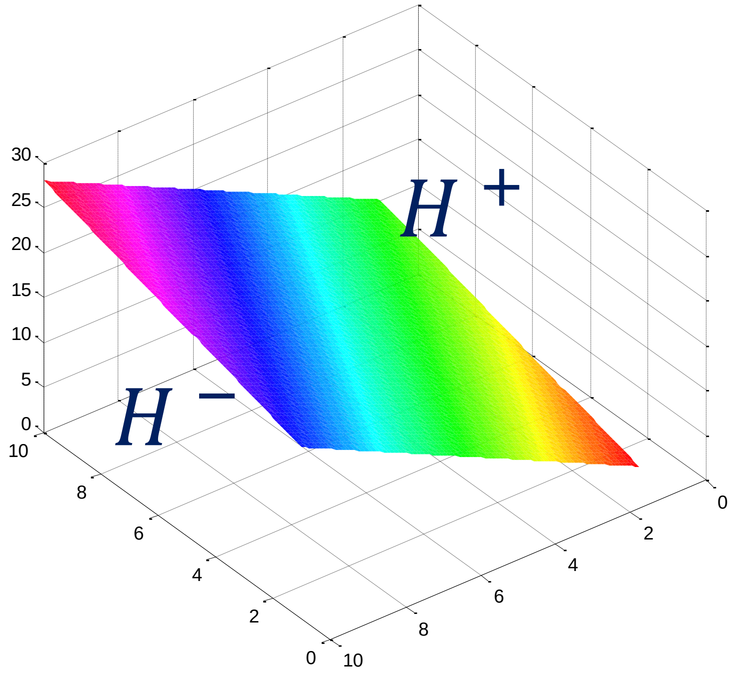
$$H(a, b) = \{X|a^TX = b\}$$

A hyperplane has $n - 1$ dimensions in an n-dimensional space

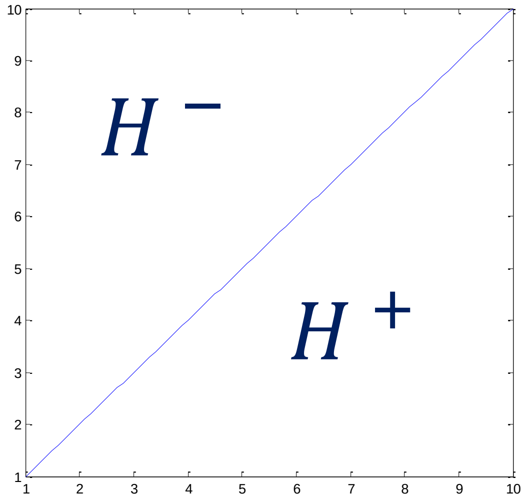
- ✓ It is a plane in three dimensional space
- ✓ It is a line in two dimensional space

Linear Problem (LP)

Some definitions



Plane



Line

$$H^+ = \{X \mid a^T X \geq b\}$$

$$H^- = \{X \mid a^T X \leq b\}$$

Some definitions

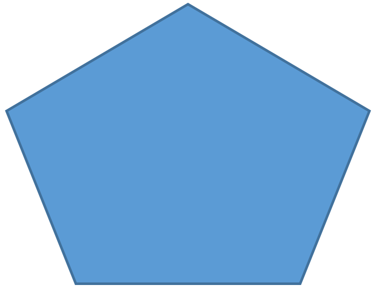
Convex Set

A convex set is a collection of points such that if X^1 and X^2 are any two points in the collection, the line segment joining them is also in the collection, which can be defined as follows

If $X^1, X^2 \in S$, then $X \in S$

Where $X(\lambda) = \lambda X^1 + (1-\lambda)X^2 \qquad 0 \le \lambda \le 1$

Vertex or Extreme point



Some definitions

Feasible solution

In a linear programming problem, any solution that satisfy the conditions

$$aX = b$$

$$X \geq 0$$

is called feasible solution

Some definitions

Basic solution

A basic solution is one in which $n - m$ variable are set equal to zero and solution can be obtained for the m number variables

$$
\begin{aligned}
2x_1 + 3x_2 - 2x_3 - 7x_4 &= 1 \\
x_1 + x_2 + x_3 + 3x_4 &= 6 \\
x_1 - x_2 + x_3 + 5x_4 &= 4
\end{aligned}
$$

Here $n = 4$ and $m = 3$, i.e. no. of variable is 4 and no. of equation is 3.

Some definitions

Basis

The collection of variables not set equal to zero to obtain the basic solution is called the basis.

$$2x_1 + 3x_2 - 2x_3 - 7x_4 = 1$$
$$x_1 + x_2 + x_3 + 3x_4 = 6$$
$$x_1 - x_2 + x_3 + 5x_4 = 4$$

Basic feasible solution

This is the basic solution that satisfies the non-negativity conditions

Nondegenerate basic feasible solution

This is a basic feasible solution that has got exactly m positive x_i

Optimal solution

A feasible solution that optimized the objective function is called an optimal solution

Linear Problem (LP)

Solution of system of linear simultaneous equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \longrightarrow \mathrm{E}_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \longrightarrow \mathrm{E}_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \longrightarrow \mathrm{E}_3$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \longrightarrow \mathrm{E}_n$$

Elementary operation

1. Any equation E_r can be replaced by kE_r, where k is a non zero constant
2. Any equation E_r can be replaced by $E_r + kE_s$, where E_s is any other equation

Linear Problem (LP)

Using these elementary row operation, a particular variable can be eliminated from all but one equation. This operation is known as Pivot operation

Using pivot operation, we can transform the set of equation to the following form

$$
\begin{array}{c}
1x_1 + 0x_2 + 0x_3 + \cdots + 0x_n = b'_1 \\
0x_1 + 1x_2 + 0x_3 + \cdots + 0x_n = b'_2 \\
0x_1 + 0x_2 + 1x_3 + \cdots + 0x_n = b'_3 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
0x_1 + 0x_2 + 0x_3 + \cdots + 1x_n = b'_n
\end{array}
$$

Now the solution are

$$x_i = b'_i \qquad i = 1,2,3, \ldots , n$$

Linear Problem (LP)

General system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_1 + a_{m3}x_1 + \cdots + a_{mn}x_n = b_m$$

And $n > m$

Pivotal variables

Non pivotal variables

Constants

$$1x_1 + 0x_2 + \cdots + 0x_m + a'_{1m+1}x_{m+1} + \cdots + a'_{1n}x_n = b'_1$$
$$0x_1 + 1x_2 + \cdots + 0x_m + a'_{2m+1}x_{m+1} + \cdots + a'_{2n}x_n = b'_2$$
$$0x_1 + 0x_2 + \cdots + 0x_m + a'_{3m+1}x_{m+1} + \cdots + a'_{3n}x_n = b'_3$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$0x_1 + 0x_2 + \cdots + 1x_m + a'_{mm+1}x_{m+1} + \cdots + a'_{mn}x_n = b'_m$$

General system of equations

$$1x_1 + 0x_2 + \cdots + 0x_m + a'_{1m+1}x_{m+1} + \cdots + a'_{1n}x_n = b'_1$$
$$0x_1 + 1x_2 + \cdots + 0x_m + a'_{2m+1}x_{m+1} + \cdots + a'_{2n}x_n = b'_2$$
$$0x_1 + 0x_2 + \cdots + 0x_m + a'_{3m+1}x_{m+1} + \cdots + a'_{3n}x_n = b'_3$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$0x_1 + 0x_2 + \cdots + 1x_m + a'_{mm+1}x_{m+1} + \cdots + a'_{mn}x_n = b'_m$$

One solution can be deduced from the system of equations are

$x_i = b'_i$ For $i = 1,2,3, \dots, m$

$x_i = 0$ For $i = m+1, m+2, m+3, \dots, n$

This solution is called basic solution

Basic variable x_i $i = 1,2,3, \dots, m$

Non basic variable x_i $i = m+1, m+2, m+3, \dots, n$

Thanks