Linear Problem (LP)

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Linear Problem (LP)

General system of equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\
a_{m1}x_1 + a_{m2}x_1 + a_{m3}x_1 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

And $n > m$

Pivotal variables

Non pivotal variables

Constants

$$
\begin{aligned}
1x_1 + 0x_2 + \cdots + 0x_m + a'_{1m+1}x_{m+1} + \cdots + a'_{1n}x_n &= b'_1 \\
0x_1 + 1x_2 + \cdots + 0x_m + a'_{2m+1}x_{m+1} + \cdots + a'_{2n}x_n &= b'_2 \\
0x_1 + 0x_2 + \cdots + 0x_m + a'_{3m+1}x_{m+1} + \cdots + a'_{3n}x_n &= b'_3 \\
\vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\
0x_1 + 0x_2 + \cdots + 1x_m + a'_{mm+1}x_{m+1} + \cdots + a'_{mn}x_n &= b'_m
\end{aligned}
$$

Linear Problem (LP)

General system of equations

$$1x_1 + 0x_2 + \cdots + 0x_m + a'_{1m+1}x_{m+1} + \cdots + a'_{1n}x_n = b'_1$$
$$0x_1 + 1x_2 + \cdots + 0x_m + a'_{2m+1}x_{m+1} + \cdots + a'_{2n}x_n = b'_2$$
$$0x_1 + 0x_2 + \cdots + 0x_m + a'_{3m+1}x_{m+1} + \cdots + a'_{3n}x_n = b'_3$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$0x_1 + 0x_2 + \cdots + 1x_m + a'_{mm+1}x_{m+1} + \cdots + a'_{mn}x_n = b'_m$$

One solution can be deduced from the system of equations are

$x_i = b'_i$ For $i = 1,2,3, \ldots, m$

$x_i = 0$ For $i = m+1, m+2, m+3, \ldots, n$

This solution is called basis solution

Basic variable x_i $i = 1,2,3, \ldots, m$

Non basic variable x_i $i = m+1, m+2, m+3, \ldots, n$

Linear Problem (LP)

General system of equations

Now let's solve a problem

$$2x_1 + 3x_2 - 2x_3 - 7x_4 = 1 \qquad R_{00}$$

$$x_1 + x_2 + x_3 + 3x_4 = 6 \qquad R_{10}$$

$$x_1 - x_2 + x_3 + 5x_4 = 4 \qquad R_{20}$$

$$x_1 + \frac{3}{2}x_2 - x_3 - \frac{7}{2}x_4 = \frac{1}{2} \qquad R_{01} = \frac{1}{2}R_{00}$$

$$0 - \frac{1}{2}x_2 + 2x_3 + \frac{13}{2}x_4 = \frac{11}{2} \qquad R_{11} = R_{10} - R_{01}$$

$$0 - \frac{5}{2}x_2 + 2x_3 + \frac{17}{2}x_4 = \frac{7}{2} \qquad R_{21} = R_{20} - R_{01}$$

Linear Problem (LP)

General system of equations

$$x_1 + 0 + 5x_3 + 16x_4 = 17 \qquad R_{02} = R_{12} - \frac{3}{2}R_{12}$$

$$0 + x_2 - 4x_3 - 13x_4 = -11 \qquad R_{12} = -2R_{11}$$

$$0 + 0 - \boxed{8x_3} - 24x_4 = -24 \qquad R_{22} = R_{12} + \frac{5}{2}R_{12}$$

$$x_1 + 0 + 0 + x_4 = 2 \qquad R_{03} = R_{02} - 5R_{23}$$

$$0 + x_2 + 0 - x_4 = 1 \qquad R_{13} = R_{12} + 4R_{23}$$

$$0 + 0 + x_3 + 3x_4 = 3 \qquad R_{23} = -\frac{1}{8}R_{22}$$

General system of equations

Solution of the problem is

$$x_1 = 2 - x_4$$

$$x_2 = 1 + x_4$$

$$x_3 = 3 - 3x_4$$

The solution obtain by setting independent variable equal to zero is called basic solution.

$$x_1 = 2 \qquad x_2 = 1 \qquad x_3 = 3$$

General system of equations

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 - 7x_4 &= 1 \\ x_1 + x_2 + x_3 + 3x_4 &= 6 \\ x_1 - x_2 + x_3 + 5x_4 &= 4 \end{aligned}$$

$x_1 = 2, x_2 = 1, x_3 = 3, x_4 = 0$

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 - 7x_4 &= 1 \\ x_1 + x_2 + x_3 + 3x_4 &= 6 \\ x_1 - x_2 + x_3 + 5x_4 &= 4 \end{aligned}$$

$x_1 = 1, x_2 = 2, x_3 = 0, x_4 = 1$

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 - 7x_4 &= 1 \\ x_1 + x_2 + x_3 + 3x_4 &= 6 \\ x_1 - x_2 + x_3 + 5x_4 &= 4 \end{aligned}$$

$x_1 = 3, x_2 = 0, x_3 = 6, x_4 = -1$

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 - 7x_4 &= 1 \\ x_1 + x_2 + x_3 + 3x_4 &= 6 \\ x_1 - x_2 + x_3 + 5x_4 &= 4 \end{aligned}$$

$x_1 = 0, x_2 = 3, x_3 = -3, x_4 = 2$

General system of equations

How many combinations?

$$\binom{n}{m} = \frac{n!}{(n-m)!\,m!}$$

The problem we have just solved has 4 combinations

Now consider a problem of 10 variables and 8 equations, we will have 45 different combinations

If a problem of 15 variables and 10 equations, we will have 3003 different combinations

As such, it is not possible to find solutions for all the combinations

Moreover, many combinations, we may get infeasible solutions

As such we need some set of rules to switch from one feasible solution another feasible solution

Linear Problem (LP)

Now before discussing any method, let's try to solve a problem

Minimize $-x_1 - 2x_2 - x_3$

Subject to

$$2x_1 + x_2 - x_3 \leq 2$$
$$2x_1 - x_2 + 5x_3 \leq 6$$
$$4x_1 + x_2 + x_3 \leq 6$$
$$x_i \geq 0 \qquad i = 1,2,3$$

$$2x_1 + x_2 - x_3 + x_4 = 2$$
$$2x_1 - x_2 + 5x_3 + x_5 = 6$$
$$4x_1 + x_2 + x_3 + x_6 = 6$$
$$-x_1 - 2x_2 - x_3 - f = 0$$

The initial basic solution is $x_4 = 2 \quad x_5 = 6 \quad x_6 = 6$ Basic variable

$x_1 = x_2 = x_3 = 0$ Non basic variable

$f = 0$

Linear Problem (LP)

Now look at the objective function

$$-x_1 - 2x_2 - x_3 \qquad\qquad -f = 0$$

Is it an optimal solution?

Can we improve the objective function value by making one non basic variable as basic?

For this problem, all the coefficients of the objective function is negative, as such making one of them as basic variable, we can improve (reduce) the objective value.

However, making x_2 as basic variable we will have maximum advantage

So, select the variable with minimum negative coefficient

Linear Problem (LP)

In our problem, x_2 is the new entering variable (basic variable)

Now, next question is which one will be pivoting element

$$
\begin{aligned}
2x_1 + x_2 - x_3 + x_4 &= 2 \\
2x_1 - x_2 + 5x_3 + x_5 &= 6 \\
4x_1 + x_2 + x_3 + x_6 &= 6 \\
-x_1 - 2x_2 - x_3 - f &= 0
\end{aligned}
$$

$$
\begin{aligned}
2x_1 + x_2 - x_3 + x_4 &= 2 \\
4x_1 + 0x_2 + 4x_3 + x_4 + x_5 &= 8 \\
2x_1 + 0x_2 + 2x_3 - x_4 + x_6 &= 4 \\
3x_1 + 0x_2 - 3x_3 + x_4 - f &= 4
\end{aligned}
$$

The initial basic solution is $x_2 = 2$ $x_5 = 8$ $x_6 = 4$ Basic variable

$x_1 = x_3 = x_4 = 0$ Non basic variable

$f = -4$

Linear Problem (LP)

$$2x_1 + x_2 - x_3 + x_4 = 2 \qquad 4x_1 + 0x_2 + 4x_3 + x_4 + x_5 = 8$$
$$2x_1 - x_2 + 5x_3 + x_5 = 6 \qquad -2x_1 + x_2 - 5x_3 - x_5 = -6$$
$$4x_1 + x_2 + x_3 + x_6 = 6 \qquad 6x_1 + 0x_2 + 6x_3 + x_5 + x_6 = 12$$
$$-x_1 - 2x_2 - x_3 - f = 0 \qquad -5x_1 + 0x_2 - 11x_3 - 2x_5 - f = -12$$

The initial basic solution is $x_2 = -6 \quad x_4 = 8 \quad x_6 = 12$ Basic variable

$x_1 = x_3 = x_5 = 0$ Non basic variable

$f = -12$

Linear Problem (LP)

$$
\begin{aligned}
2x_1 + x_2 - x_3 + x_4 &= 2 \\
2x_1 - x_2 + 5x_3 + x_5 &= 6 \\
4x_1 + x_2 + x_3 + x_6 &= 6 \\
-x_1 - 2x_2 - x_3 - f &= 0
\end{aligned}
$$

$$
\begin{aligned}
-2x_1 + 0x_2 - 2x_3 + x_4 - x_6 &= -4 \\
6x_1 + 0x_2 + 6x_3 + x_5 + x_6 &= 12 \\
4x_1 + x_2 + x_3 + x_6 &= 6 \\
7x_1 + 0x_2 + x_3 + 2x_6 - f &= 12
\end{aligned}
$$

The initial basic solution is $x_2 = 6$ $x_4 = -4$ $x_5 = 12$ Basic variable

$x_1 = x_3 = x_6 = 0$ Non basic variable

$f = +12$

Linear Problem (LP)

$$2x_1 + x_2 - x_3 + x_4 = 2$$
$$2x_1 - x_2 + 5x_3 + x_5 = 6$$
$$4x_1 + x_2 + x_3 + x_6 = 6$$
$$-x_1 - 2x_2 - x_3 - f = 0$$

$x_2 = 2 \quad x_5 = 8 \quad x_6 = 4$

$x_1 = x_3 = x_4 = 0$

$f = -4$

$x_2 = -6 \quad x_4 = 8 \quad x_6 = 12$

$x_1 = x_3 = x_5 = 0$

$f = -12$

Infeasible solution

$x_2 = 6 \quad x_4 = -4 \quad x_5 = 12$

$x_1 = x_3 = x_6 = 0$

$f = +12$

Infeasible solution

Now what is the rule, how to select the pivoting element?

Linear Problem (LP)

What is the maximum value of x_2 without making the solution negative?

$$2x_1 + x_2 - x_3 + x_4 = 2 \qquad x_2 = 2/1$$

$$2x_1 - x_2 + 5x_3 + x_5 = 6$$

$$4x_1 + x_2 + x_3 + x_6 = 6 \qquad x_2 = 6/1$$

Select the minimum one to avoid infeasible solution

Thus the general rule is

1. Calculate the ratio $\frac{b_i}{a_{is}}$ (For $a_{is} \geq 0$)
2. Pivoting element is $x_s^* = \underset{a_{is} \geq 0}{minimum} \left(\frac{b_i}{a_{is}}\right)$

Linear Problem (LP)

$$2x_1 + x_2 - x_3 + x_4 = 2$$
$$2x_1 - x_2 + 5x_3 + x_5 = 6$$
$$4x_1 + x_2 + x_3 + x_6 = 6$$
$$-x_1 - 2x_2 - x_3 - f = 0$$

Basic Variable	Variable						f	bi	bi/aij
	x1	x2	x3	x4	x5	x6			
x4	2	1	-1	1	0	0	0	2	2
x5	2	-1	5	0	1	0	0	6	
x6	4	1	1	0	0	1	0	6	6
f	-1	-2	-1	0	0	0	-1	0	

Linear Problem (LP)

Basic Variable	Variable						f	bi	bi/aij
	x1	x2	x3	x4	x5	x6			
x2	2	1	-1	1	0	0	0	2	
x5	4	0	4	1	1	0	0	8	2
x6	2	0	2	-1	0	1	0	4	2
f	3	0	-3	2	0	0	-1	4	

Basic Variable	Variable						f	bi	bi/aij
	x1	x2	x3	x4	x5	x6			
x2	3	1	0	1.25	0.25	0	0	4	
x3	1	0	1	0.25	0.25	0	0	2	
x6	0	0	0	-1.5	-0.5	1	0	0	
f	6	0	0	2.75	0.75	0	-1	10	

All c_j are positive, so no improvement is possible

Linear Problem (LP)

Basic Variable	Variable						f	bi	bi/aij
	x1	x2	x3	x4	x5	x6			
x2	2	1	-1	1	0	0	0	2	
x5	4	0	4	1	1	0	0	8	2
x6	2	0	2	-1	0	1	0	4	2
f	3	0	-3	2	0	0	-1	4	

Basic Variable	Variable						f	bi	bi/aij
	x1	x2	x3	x4	x5	x6			
x2	3	1	0	0.5	0	0.5	0	4	
x5	0	0	0	3	1	-2	0	0	
x3	1	0	1	-0.5	0	0.5	0	2	
f	6	0	0	0.5	0	1.5	-1	10	

Obtain the same solution

Linear Problem (LP)

Thanks