Linear Problem: SIMPLEX METHOD

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General system of equations

$$Minimize\ f = c_1x_1 + c_2x_2 + \cdots + c_mx_m$$

$$Subject\ to$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_1 + a_{m3}x_1 + \cdots + a_{mn}x_n = b_m$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_1 + a_{m3}x_1 + \cdots + a_{mn}x_n = b_m$$

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n - f = -f_0$$

Linear Problem (LP)

SIMPLEX METHOD

$$1x_1 + 0x_2 + \cdots + 0x_m + a'_{1m+1}x_{m+1} + \cdots + a'_{1n}x_n = b'_1$$
$$0x_1 + 1x_2 + \cdots + 0x_m + a'_{2m+1}x_{m+1} + \cdots + a'_{2n}x_n = b'_2$$
$$0x_1 + 0x_2 + \cdots + 0x_m + a'_{3m+1}x_{m+1} + \cdots + a'_{3n}x_n = b'_3$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$0x_1 + 0x_2 + \cdots + 1x_m + a'_{mm+1}x_{m+1} + \cdots + a'_{mn}x_n = b'_m$$

$$0x_1 + 0x_2 + \cdots + 0x_m - f + c'_{m+1}x_{m+1} + \cdots + c'_n x_n = -f'_o$$

$x_i = b'_i$ For $i = 1,2,3, \ldots, m$

$x_i = 0$ For $i = m+1, m+2, m+3, \ldots, n$

$f = f'_o$

If the basic solution is feasible , then $b'_i \geq 0$ for $i = 1,2,3, \ldots, m$

Linear Problem (LP)

From the last row

$$0x_1 + 0x_2 + \cdots + 0x_m - f + c'_{m+1}x_{m+1} + \cdots + c'_n x_n = -f'_o$$

We can write that

$$f = f'_o + \sum_{i=m+1}^{n} c'_i x_i$$

If all c'_i are positive, it is not possible to improve (reduce) the objective function value by making a non basic variable as basic variable

Maximum benefit can be obtained by making the non-basic variable with minimum negative coefficient as basic variable

In case of a tie, any one can be selected arbitrarily

Linear Problem (LP)

$$x_1 = b_1' - a_{1s}' x_s \qquad b_1' \geq 0$$

$$x_2 = b_2' - a_{2s}' x_s \qquad b_2' \geq 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_m = b_m' - a_{ms}' x_s \qquad b_m' \geq 0$$

If a_{is}' is positive, the maximum possible value of x_s is b_i'/a_{is}'

If a_{is}' is negative, the maximum possible value of x_s is $+\infty$

In this case, the problem has an unbounded solution

Linear Problem (LP)

Example 1 (Unbounded solution)

Minimize $\text{f} = -3x_1 - 2x_2$

Subject to

$$x_1 - x_2 \leq 1$$
$$3x_1 - 2x_2 \leq 6$$
$$x_i \geq 0 \qquad i = 1,2,3$$

$$x_1 - x_2 + x_3 = 1$$
$$3x_1 - 2x_2 + x_4 = 6$$
$$x_i \geq 0 \qquad i = 1,2,3$$

Basic Variable	Variable				f	bi	bi/aij
	x1	x2	x3	x4			
x3	1	-1	1	0	0	1	1
x4	3	-2	0	1	0	6	2
f	-3	-2	0	0	-1	0	

Linear Problem (LP)

Basic Variable	Variable				f	bi	bi/ais
	x1	x2	x3	x4			
x1	1	-1	1	0	0	1	
x4	0	1	-3	1	0	3	3
f	0	-5	3	0	-1	3	

Basic Variable	Variable				f	bi	bi/ais
	x1	x2	x3	x4			
x1	1	0	-2	1	0	4	
x2	0	1	-3	1	0	3	
f	0	0	-12	5	-1	18	

All a_{ij} are negative

Unbounded solution

Linear Problem (LP)

Example 2 (Alternate optimal solutions)

Minimize $f = -40x_1 - 100x_2$

Subject to

$$10x_1 + 5x_2 \leq 2500$$
$$4x_1 + 10x_2 \leq 2000$$
$$2x_1 + 3x_2 \leq 900$$
$$x_i \geq 0 \qquad i = 1,2,3$$

$$10x_1 + 5x_2 + x_3 = 2500$$
$$4x_1 + 10x_2 + x_4 = 2000$$
$$2x_1 + 3x_2 + x_5 = 900$$
$$x_i \geq 0 \qquad i = 1,2,3,4,5$$

Linear Problem (LP)

Basic Variable	Variable							
	x1	x2	x3	x4	x5	f	b	bi/ais
x3	10	5	1	0	0	0	2500	500
x4	4	10	0	1	0	0	2000	200
x5	2	3	0	0	1	0	900	300
f	-40	-100	0	0	0	-1	0	

Basic Variable	Variable							
	x1	x2	x3	x4	x5	f	b	bi/ais
x3	8	0	1	-0.5	0	0	1500	187.5
x2	0.4	1	0	0.1	0	0	200	500
x5	0.8	0	0	-0.3	1	0	300	375
f	0	0	0	10	0	-1	20000	

All c_j are positive, so no improvement is possible

Solution is

$x_3 = 1500$

$x_2 = 200$

$x_5 = 300$

$x_1 = x_4 = 0$

$f = -20000$

Linear Problem (LP)

Basic Variable	Variable							bi/ais
	x1	x2	x3	x4	x5	f	b	
x1	1	0	0.125	-0.0625	0	0	187.5	
x2	0	1	-0.05	0.125	0	0	125	
x5	0	0	-0.1	-0.25	1	0	150	
f	0	0	0	10	0	-1	20000	

Solution is

$x_1 = 187.5$

$x_2 = 125$

$x_5 = 150$

$x_3 = x_4 = 0$

$f = -20000$

The problem has infinite number of optimal solutions, which can be obtained using the following equation

$$X(\lambda) = \lambda X^1 + (1 - \lambda)X^2$$

Linear Problem (LP)

Example 3 (Artificial variable)

Minimize $f = 2x_1 + 3x_2 + 2x_3 - x_4 + x_5$

Subject to

$$3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 = 0$$
$$x_1 + x_2 + x_3 + 3x_4 + x_5 = 2$$

$$x_i \geq 0 \qquad i = 1,2,\ldots,5$$

$$3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 + y_1 = 0$$
$$x_1 + x_2 + x_3 + 3x_4 + x_5 + y_2 = 2$$

$$x_i \geq 0 \qquad i = 1,2,\ldots,5$$
$$y_1, y_2 \geq 0$$

y_1 and y_2 Artificial variable

Linear Problem (LP)

$$
\begin{aligned}
3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 + y_1 \quad &= 0 \\
x_1 + x_2 + x_3 + 3x_4 + x_5 \quad + y_2 \quad &= 2 \\
2x_1 + 3x_2 + 2x_3 - x_4 + x_5 \quad - f &= 0
\end{aligned}
$$

The Artificial variables have to be removed from the basis initially (Phase I)

This can be remove using the following formulation

Minimize $w = y_1 + y_2$

Now the problem

$$
\begin{aligned}
3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 + y_1 \quad &= 0 \\
x_1 + x_2 + x_3 + 3x_4 + x_5 \quad + y_2 \quad &= 2 \\
2x_1 + 3x_2 + 2x_3 - x_4 + x_5 \quad - f \quad &= 0 \\
y_1 + y_2 - w \quad &= 0
\end{aligned}
$$

Linear Problem (LP)

$$3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 + y_1 = 0$$

$$x_1 + x_2 + x_3 + 3x_4 + x_5 + y_2 = 2$$

$$2x_1 + 3x_2 + 2x_3 - x_4 + x_5 - f = 0$$

$$-4x_1 + 2x_2 - 5x_3 - 5x_4 + 0x_5 - w = -2$$

Basic Variable	Variable										
	x1	x2	x3	x4	x5	y1	y2	f	w	b	bi/ais
y1	3	-3	4	2	-1	1	0	0	0	0	0
y2	1	1	1	3	1	0	1	0	0	2	0.67
f	2	3	2	-1	1	0	0	-1	0	0	
w	-4	2	-5	-5	0	0	0	0	-1	-2	

Linear Problem (LP)

Basic Variable	Variable										
	x1	x2	x3	x4	x5	y1	y2	f	w	b	bi/ais
x4	1.5	-1.5	2	1	-0.5	0.5	0	0	0	0	
y2	-3.5	5.5	-5	0	2.5	-1.5	1	0	0	2	0.36
f	3.5	1.5	4	0	0.5	0.5	0	-1	0	0	
w	3.5	-5.5	5	0	-2.5	2.5	0	0	-1	-2	

Basic Variable	Variable										
	x1	x2	x3	x4	x5	y1	y2	f	w	b	bi/ais
x4	0.55	0	0.64	1	0.18	0.09	0.27	0	0	0.55	
x2	-0.64	1	-0.91	0	0.45	-0.27	0.18	0	0	0.36	
f	4.45	0	5.36	0	-0.18	0.91	-0.27	-1	0	-0.55	
w	0	0	0	0	0	1	1	0	-1	0	

All c_j are positive, so no improvement is possible

Linear Problem (LP)

Phase II

Basic Variable	Variable							bi/ais
	x1	x2	x3	x4	x5	f	b	
x4	0.55	0	0.64	1	0.18	0	0.55	3
x2	-0.64	1	-0.91	0	0.45	0	0.36	0.8
f	4.45	0	5.36	0	-0.18	-1	-0.55	

Basic Variable	Variable							bi/ais
	x1	x2	x3	x4	x5	f	b	
x4	0.8	-0.4	1	1	0	0	0.4	
x5	-1.4	2.2	-2	0	1	0	0.8	
f	4.2	0.4	5	0	0	-1	-0.4	

Solution is

$x_4 = 0.4$

$x_5 = 0.8$

$x_1 = x_2 = x_1 = 0$

$f = 0.4$

All c_j are positive, so no improvement is possible

Optimal solution

Example 4 (Unrestricted in sign)

Minimize $f = 4x_1 + 2x_2$

Subject to

$$x_1 - 2x_2 \geq 2$$
$$x_1 + 2x_2 = 8$$
$$x_1 - x_2 \leq 11$$
$$x_1 \geq 0$$

x_2 is unrestricted in sign

Consider $x_2 = x_3 - x_4$

Where, $x_3, x_4 \geq 0$

Now, the problem can be written as

Minimize $f = 4x_1 + 2x_3 - 2x_4$

Subject to

$$x_1 - 2x_3 + 2x_4 \geq 2$$
$$x_1 + 2x_3 - 2x_4 = 8$$
$$x_1 - x_3 + x_4 \leq 11$$
$$x_i \geq 0 \qquad i = 1,3,4$$

Linear Problem (LP)

$$x_1 - 2x_3 + 2x_4 - x_5 + y_1 = 2$$
$$x_1 + 2x_3 - 2x_4 + y_2 = 8$$
$$x_1 - x_3 + x_4 + x_6 = 11$$
$$4x_1 + 2x_3 - 2x_4 - f = 0$$

Phase I

Minimize $w = y_1 + y_2$

Or, Minimize $\mathrm{w} = -2x_1 + 0x_3 + 0x_4 + x_5 = -10$

Linear Problem (LP)

Phase I problem can be written as

$$x_1 - 2x_3 + 2x_4 - x_5 + y_1 = 2$$

$$x_1 + 2x_3 - 2x_4 + y_2 = 8$$

$$x_1 - x_3 + x_4 + x_6 = 11$$

$$4x_1 + 2x_3 - 2x_4 - f = 0$$

$$-2x_1 + 0x_3 + 0x_4 + x_5 - w = -10$$

Basic Variable	Variable							w	f	bi	bi/aij
	x1	x3	x4	x5	x6	y1	y2				
y1	1	-2	2	-1	0	1	0	0	0	2	2
y2	1	2	-2	0	0	0	1	0	0	8	8
x6	1	-1	1	0	1	0	0	0	0	11	11
f	4	2	-2	0	0	0	0	0	-1	0	
w	-2	0	0	1	0	0	0	-1	0	-10	

Linear Problem (LP)

Basic Variable	Variable							w	f	bi	bi/aij
	x1	x3	x4	x5	x6	y1	y2				
x1	1	-2	2	-1	0	1	0	0	0	2	
y2	0	4	-4	1	0	-1	1	0	0	6	1.5
x6	0	1	-1	1	1	-1	0	0	0	9	9
f	0	10	-10	4	0	-4	0	0	-1	-8	
w	0	-4	4	-1	0	2	0	-1	0	-6	

Basic Variable	Variable							w	f	bi	bi/aij
	x1	x3	x4	x5	x6	y1	y2				
x1	1	0	0	-0.5	0	0.5	0.5	0	0	5	
x3	0	1	-1	0.25	0	-0.25	0.25	0	0	1.5	
x6	0	0	0	0.75	1	-0.75	-0.25	0	0	7.5	
f	0	0	0	1.5	0	-1.5	-2.5	0	-1	-23	
w	0	0	0	0	0	1	1	-1	0	0	

All c_j are positive, so no improvement is possible

Linear Problem (LP)

Phase II

Basic Variable		Variable				f	bi	bi/aij
	x1	x3	x4	x5	x6			
x1	1	0	0	-0.50	0	0	5	
x3	0	1	-1	0.25	0	0	1.5	
x6	0	0	0	0.75	1	0	7.5	
f	0	0	0	1.5	0	-1	-23	

It can be noted that all the coefficients of the cost function is positive, hence it is not possible to improve the objective function value

This the optimal solution of the problem is

$x_1 = 5 \quad x_3 = 1.5 \qquad x_6 = 7.5 \qquad x_4 = x_5 = 0 \qquad f = 23$

$x_2 = x_3 - x_4 = 1.5$

Example 5

A manufacturer produces, A, B, C, and D, by using two types of machines (lathes and milling machines). The time required on the two machines to manufacture one unit of each of the four products, the profit per unit products and the total time available on the two types of machines per day are given below.

Machine	Time required per unit (min) for product				Available time (min)
	A	B	C	D	
Lathe machine	7	10	4	9	1200
Milling machine	3	40	1	1	800
Profit per unit	45	100	30	50	

Find the number of units to be manufactured of each product per day for maximizing profit.

Linear Problem (LP)

LP Formulation

Maximize $f = 45x_1 + 100x_2 + 30x_3 + 50x_4$

Subject to

$$7x_1 + 10x_2 + 4x_3 + 9x_4 \leq 1200$$

$$3x_1 + 40x_2 + x_3 + x_4 \leq 800$$

$$x_i \geq 0 \quad i = 1,2,3,4$$

Minimize $f = -45x_1 - 100x_2 - 30x_3 - 50x_4$

Subject to

$$7x_1 + 10x_2 + 4x_3 + 9x_4 + x_5 = 1200$$

$$3x_1 + 40x_2 + x_3 + x_4 + x_6 = 800$$

$$x_i \geq 0 \quad i = 1,2,3,4,5,6$$

Linear Problem (LP)

Basic Variable	Variable						f	bi	bi/aij
	x1	x2	x3	x4	x5	x6			
x5	7	10	4	9	1	0	0	1200	120
x6	3	40	1	1	0	1	0	800	20
f	-45	-100	-30	-50	0	0	-1	0	

Basic Variable	Variable						f	bi	bi/aij
	x1	x2	x3	x4	x5	x6			
x5	6.25	0	3.75	8.75	1	-0.25	0	1000	114
x2	0.075	1	0.025	0.025	0	0.025	0	20	800
f	-37.5	0	-27.5	-47.5	0	2.5	-1	2000	

Linear Problem (LP)

Basic Variable	Variable						f	bi	bi/aij
	x1	x2	x3	x4	x5	x6			
x4	0.71	0	0.43	1	0.11	-0.03	0	114	266
x2	0.06	1	0.01	0	0.00	0.03	0	17	1200
f	-3.57	0	-7.14	0	5.43	1.14	-1	7428	

Basic Variable	Variable						f	bi	bi/aij
	x1	x2	x3	x4	x5	x6			
x3	1.67	0	1	2.33	0.27	-0.07	0	267	
x2	0.03	1	0	-0.03	-0.01	0.03	0	13	
f	8.33	0	0	16.67	7.33	0.67	-1	9333	

This the optimal solution of the problem is

$x_1 = 0 \quad x_2 = 13 \quad x_3 = 267 \quad x_4 = 0 \quad x_5 = 0 \quad x_6 = 0 \quad f = -9333$

Linear Problem (LP)

Thanks