Constrained Optimization

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Constrained Optimization

General formulation

Min/Max $f(X)$ Where $X = [x_1, x_2, x_3, \dots, x_n]^T$

Subject to $g_j(X) = 0$ $j = 1,2,3, \dots, m$

$h_k(X) \leq 0$ $k = 1,2,3, \dots, n$

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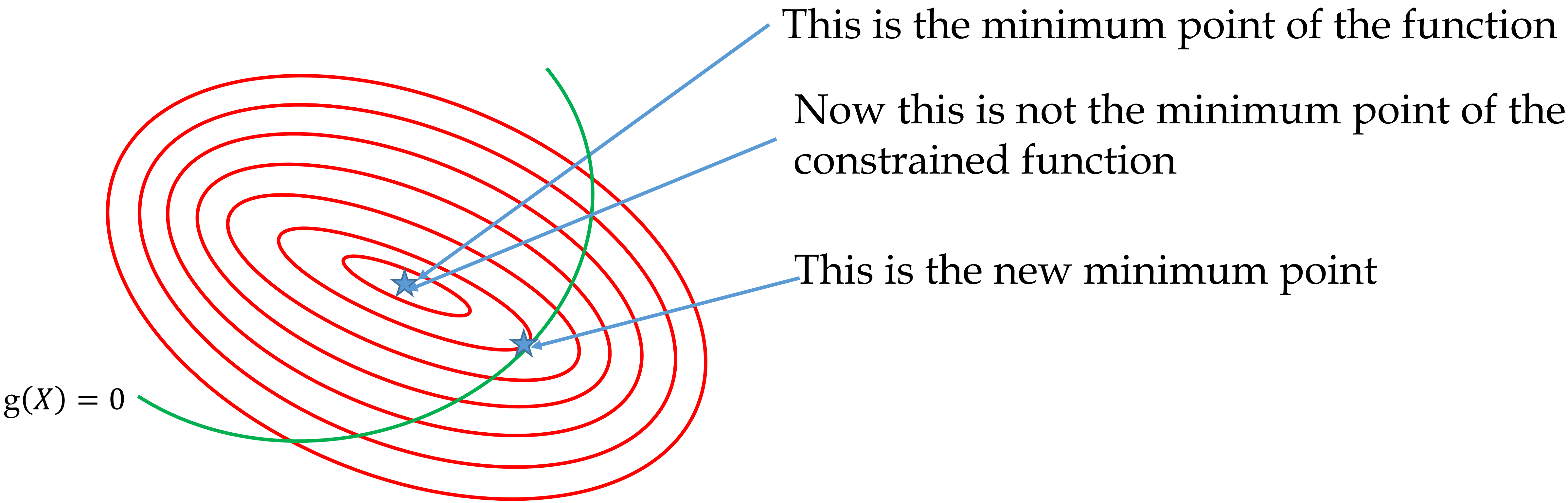
Subject to $g_j(X) = 0$ $j = 1,2,3, \dots, m$

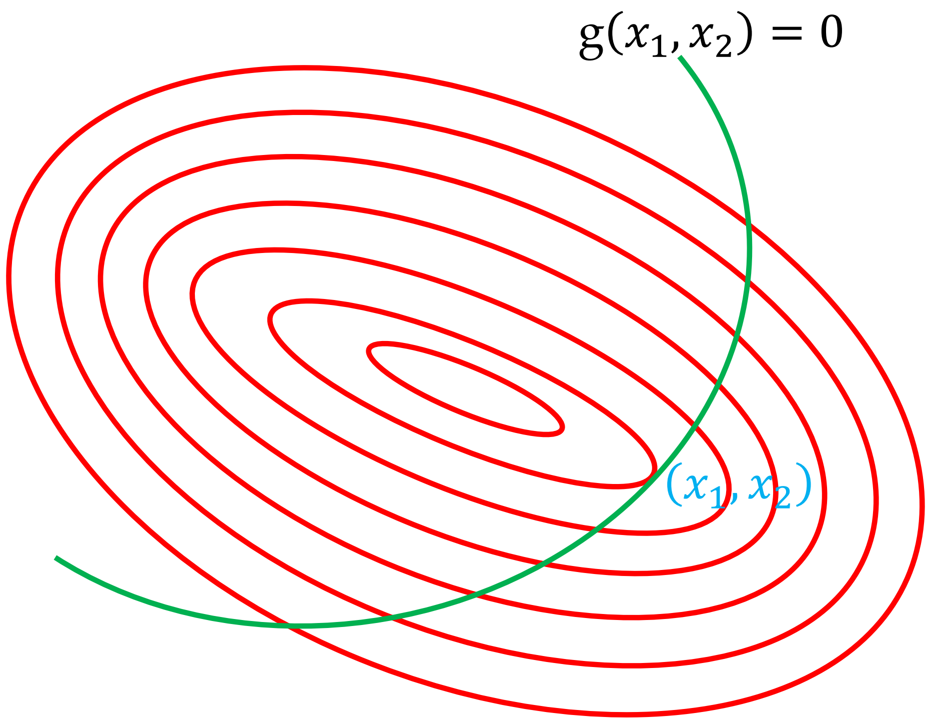


Constrained Optimization

Consider a two variables problem

Min/Max $\quad f(x_1, x_2)$

Subject to $\quad g(x_1, x_2) = 0$



Take total derivative of the function at (x_1, x_2)

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0 \quad \text{at } (x_1^*, x_2^*)$$

If (x_1, x_2) is the solution of the constrained problem, then

$$g(x_1^*, x_2^*) = 0$$

Now any variation dx_1 and dx_2 is admissible only when

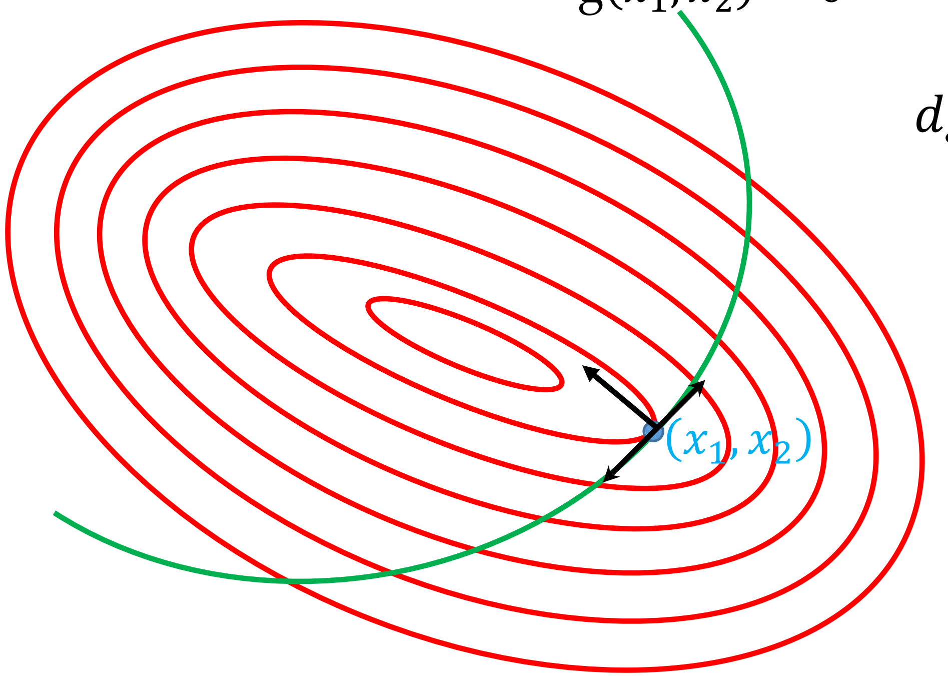
$$g(x_1^* + dx_1, x_2^* + dx_2) = 0$$

Constrained Optimization

Consider a two variables problem

Min/Max $f(x_1, x_2)$

Subject to $g(x_1, x_2) = 0$



$$g(x_1^* + dx_1, x_2^* + dx_2) = 0$$

This can be expanded as

$$g(x_1^* + dx_1, x_2^* + dx_2) = \underbrace{g(x_1^*, x_2^*)}_{0} + \frac{\partial g(x_1^*, x_2^*)}{\partial x_1} dx_1 + \frac{\partial g(x_1^*, x_2^*)}{\partial x_2} dx_2 = 0$$

$$dg = \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 = 0$$

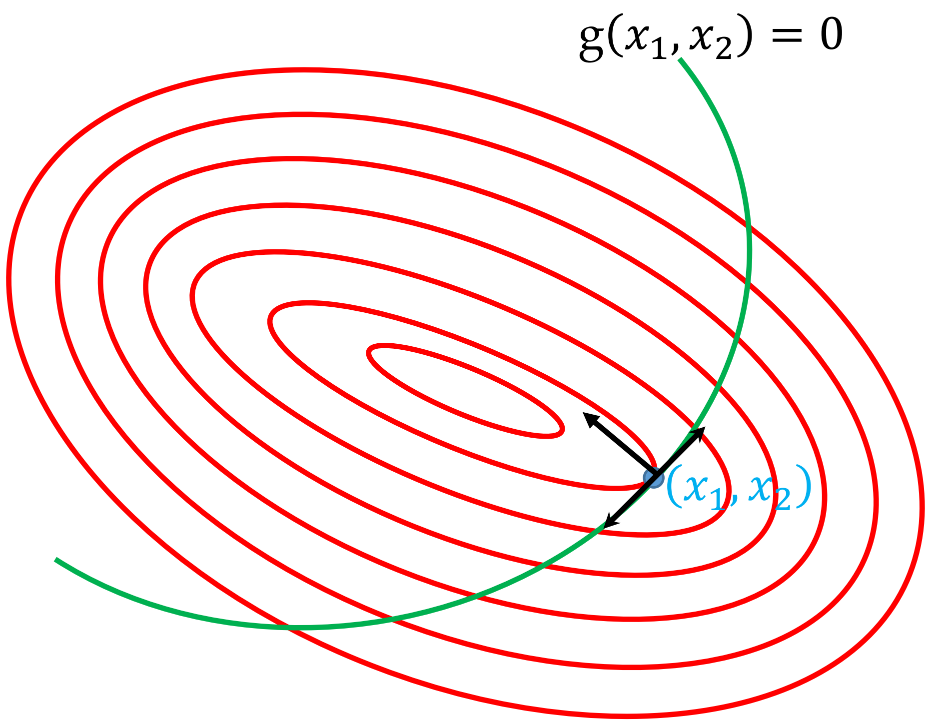
$$dx_2 = -\frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}} dx_1$$

Constrained Optimization

Consider a two variables problem

Min/Max $\quad f(x_1, x_2)$

Subject to $\quad g(x_1, x_2) = 0$



$$dx_2 = -\frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}} dx_1$$

Putting in $\quad df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0$

$$df = \frac{\partial f}{\partial x_1} dx_1 - \frac{\partial f}{\partial x_2} \frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}} dx_1 = 0$$

$$\left(\frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_1} \right) dx_1 = 0$$

$$\left(\frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_1} \right) = 0$$

This is the necessary condition for optimality for optimization problem with equality constraints

Constrained Optimization

Lagrange Multipliers

Min/Max $f(x_1, x_2)$

Subject to $g(x_1, x_2) = 0$

We have already obtained the condition that

$$\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial x_1} - \left(\frac{\frac{\partial f}{\partial x_2}}{\frac{\partial g}{\partial x_2}}\right)\frac{\partial g}{\partial x_1} = 0$$

By defining $\lambda = -\frac{\frac{\partial f}{\partial x_2}}{\frac{\partial g}{\partial x_2}}$

We have $\frac{\partial f}{\partial x_1} + \lambda\frac{\partial g}{\partial x_1} = 0$

We can also write $\frac{\partial f}{\partial x_2} + \lambda\frac{\partial g}{\partial x_2} = 0$

Also put $g(x_1, x_2) = 0$

} Necessary conditions for optimality

Constrained Optimization

Lagrange Multipliers

Let us define

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

By applying necessary condition of optimality, we can obtain

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= \frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} = 0 \\ \frac{\partial L}{\partial x_2} &= \frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2} = 0 \\ \frac{\partial L}{\partial \lambda} &= g(x_1, x_2) = 0 \end{aligned}\right\}$$

Necessary conditions for optimality

Constrained Optimization

Lagrange Multipliers

Necessary conditions for general problem

Min/Max $f(X)$ Where $X = [x_1, x_2, x_3, \dots, x_n]^T$

Subject to $g_j(X) = 0$ $j = 1,2,3, \dots, m$

$$L(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \lambda_3 \dots, \lambda_m) = f(X) + \lambda_1 g_1(X) + \lambda_2 g_2(X), \dots, \lambda_m g_m(X)$$

Necessary conditions

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0$$

$$\frac{\partial L}{\partial \lambda_j} = g_j(X) = 0$$

Constrained Optimization

Lagrange Multipliers

Min/Max $f(X)$ Where $X = [x_1, x_2, x_3, \ldots, x_n]^T$

Subject to $g(X) = b$ Or, $b - g(X) = 0$

Applying necessary conditions

$$\frac{\partial f}{\partial x_i} - \lambda \frac{\partial g}{\partial x_i} = 0$$ Where, $i = 1,2,3, \ldots, n$

$$b - g = 0$$

$$\frac{\partial g}{\partial x_i} = \frac{\frac{\partial f}{\partial x_i}}{\lambda}$$

Further

$$db - dg = 0$$

$$db = dg = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} dx_i$$

$$db = \sum_{i=1}^{n} \frac{1}{\lambda} \frac{\partial f}{\partial x_i} dx_i$$

$$db = \frac{df}{\lambda}$$

$$\lambda = \frac{df}{db}$$

$$df = \lambda db$$

<u>There may be three conditions</u>

$\lambda^* > 0$
$\lambda^* < 0$
$\lambda^* = 0$

Constrained Optimization

Find the maximum of the function $f(X) = 2x_1 + x_2 + 10$ subject to $g(X) = x_1 + 2x_2^2 = 3$ using Lagrange multiplier method. Also find the effect of changing the right-hand side of the constraint on the optimum value of f

Solution: