Constrained Optimization

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Constrained Optimization

Multivariable problem with inequality constraints

Minimize $f(X)$ Where $X = [x_1, x_2, x_3, \dots, x_n]^T$

Subject to $g_j(X) \leq 0$ $j = 1,2,3, \dots, m$

We can write $g_j(X) + y_j^2 = 0$

Thus the problem can be written as

Minimize $f(X)$

Subject to $G_j(X, Y) = g_j(X) + y_j^2 = 0$ $j = 1,2,3, \dots, m$

Where $Y = [y_1, y_2, y_3, \dots, y_m]^T$

Constrained Optimization

Multivariable problem with inequality constraints

Minimize $f(X)$ Where $X = [x_1, x_2, x_3, \dots, x_n]^T$

Subject to $G_j(X, Y) = g_j(X) + y_j^2 = 0 \qquad j = 1,2,3, \dots, m$

The Lagrange function can be written as

$$L(X, Y, \lambda) = f(X) + \sum_{j=1}^{m} \lambda_j G_j(X, Y)$$

The necessary conditions of optimality can be written as

$$\frac{\partial L(X, Y, \lambda)}{\partial x_i} = \frac{\partial f(X)}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(X)}{\partial x_i} = 0 \qquad i = 1,2,3, \dots, n$$

$$\frac{\partial L(X, Y, \lambda)}{\partial \lambda_j} = G_j(X, Y) = g_j(X) + y_j^2 = 0 \qquad j = 1,2,3, \dots, m$$

$$\frac{\partial L(X, Y, \lambda)}{\partial y_j} = 2\lambda_j y_j = 0 \qquad j = 1,2,3, \dots, m$$

Constrained Optimization

Multivariable problem with inequality constraints

From equation $\frac{\partial L(X,Y,\lambda)}{\partial y_j} = 2\lambda_j y_j = 0$

Either $\lambda_j = 0$ Or, $y_j = 0$

If $\lambda_j = 0$, the constraint is not active, hence can be ignored

If $y_j = 0$, the constraint is active, hence have to consider

Now, consider all the active constraints,
Say set J_1 is the active constraints
And set J_2 is the inactive constraints

The optimality condition can be written as

$$\frac{\partial f(X)}{\partial x_i} + \sum_{j \in J_1} \lambda_j \frac{\partial g_j(X)}{\partial x_i} = 0 \qquad i = 1,2,3,\ldots,n$$

$$g_j(X) = 0 \qquad j \in J_1$$

$$g_j(X) + y_j^2 = 0 \qquad j \in J_2$$

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Multivariable problem with inequality constraints

$$-\frac{\partial f}{\partial x_i} = \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} + \lambda_3 \frac{\partial g_3}{\partial x_i} + \cdots + \lambda_p \frac{\partial g_p}{\partial x_i} \qquad i = 1,2,3,\ldots,n$$

$$-\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \lambda_3 \nabla g_3 + \cdots + \lambda_p \nabla g_p$$

$$\nabla f = \begin{Bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \\ \vdots \\ \partial f / \partial x_n \end{Bmatrix} \qquad \nabla g_j = \begin{Bmatrix} \partial g_j / \partial x_1 \\ \partial g_j / \partial x_2 \\ \vdots \\ \partial g_j / \partial x_n \end{Bmatrix}$$

This indicates that negative of the gradient of the objective function can be expressed as a linear combination of the gradient of the active constraints at optimal point.

$$-\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

Let S be a feasible direction, then we can write

$$-S^T \nabla f = \lambda_1 S^T \nabla g_1 + \lambda_2 S^T \nabla g_2$$

Since S is a feasible direction

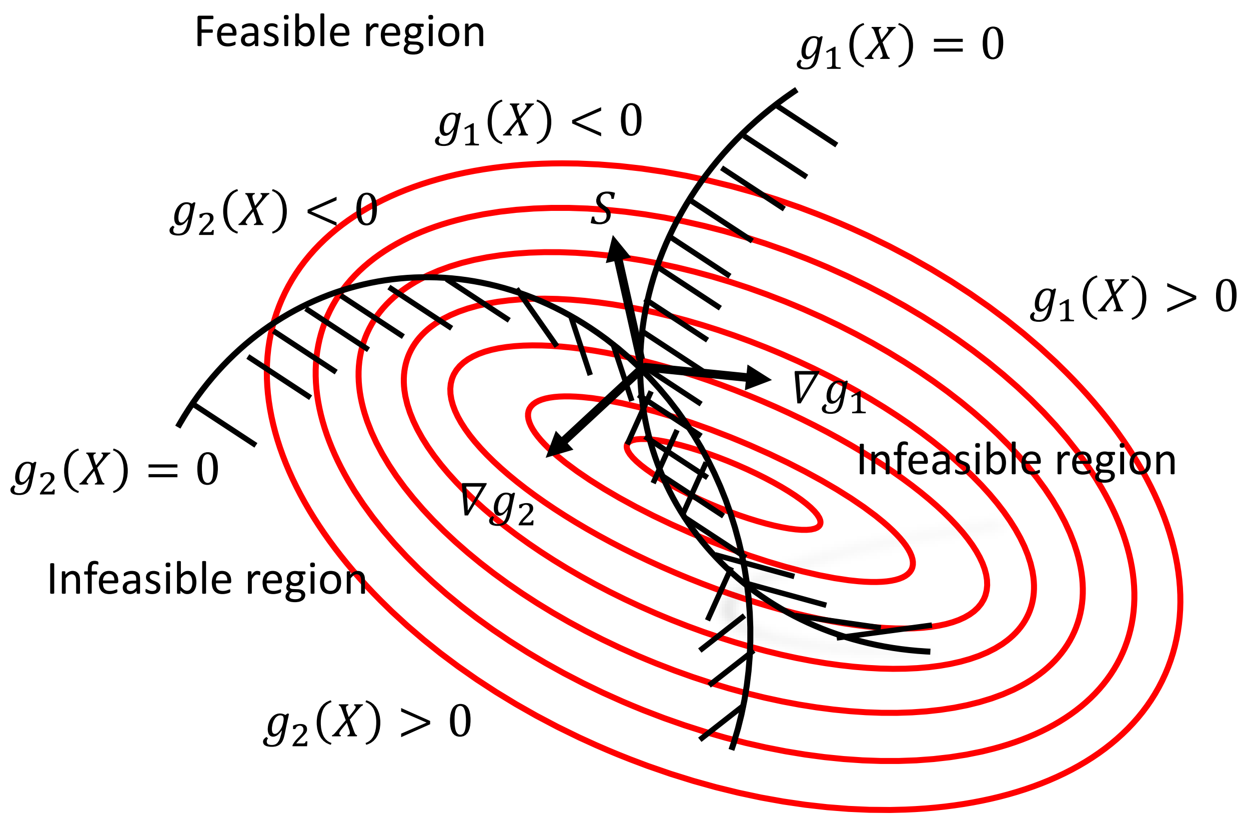
$$S^T \nabla g_1 < 0 \quad \text{and} \quad S^T \nabla g_2 < 0$$

If $\lambda_1, \lambda_2 > 0$
Then the term $S^T \nabla f$ is +ve

This indicates that S is a direction of increasing function value

Thus we can conclude that if $\lambda_1, \lambda_2 > 0$, we will not get any better solution than the current solution

Constrained Optimization



$$-S^T \nabla f = \lambda_1 S^T \nabla g_1 + \lambda_2 S^T \nabla g_2$$

Since S is a feasible direction
$S^T \nabla g_1 < 0$ and $S^T \nabla g_2 < 0$

If $\lambda_1, \lambda_2 > 0$
Then the term $S^T \nabla f$ is +ve

This indicates that S is a direction of increasing function value

Thus we can conclude that if $\lambda_1, \lambda_2 > 0$, we will not get any better solution than the current solution

Constrained Optimization

Multivariable problem with inequality constraints

The necessary conditions to be satisfied at constrained minimum points X^* are

$$\frac{\partial f(X)}{\partial x_i} + \sum_{j \in J_1} \lambda_j \frac{\partial g_j(X)}{\partial x_i} = 0 \qquad i = 1,2,3,\ldots,n$$

$$\lambda_j \geq 0 \qquad j \in J_1$$

These conditions are called **Kuhn-Tucker conditions**, the necessary conditions to be satisfied at a relative minimum of $f(X)$.

These conditions are in general not sufficient to ensure a relative minimum, However, in case of a convex problem, these conditions are the necessary and sufficient conditions for global minimum.

Constrained Optimization

Multivariable problem with inequality constraints

If the set of active constraints are not known, the Kuhn-Tucker conditions can be stated as

$$\frac{\partial f(X)}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(X)}{\partial x_i} = 0 \qquad i = 1,2,3,\dots,n$$

$$\left.\begin{aligned} \lambda_j g_j &= 0 \\ g_j &\le 0 \\ \lambda_j &\ge 0 \end{aligned}\right\} \qquad j = 1,2,3,\dots,m$$

Constrained Optimization

Multivariable problem with equality and inequality constraints

For the problem

Minimize $f(X)$ Where $X = [x_1, x_2, x_3, \dots, x_n]^T$

Subject to $g_j(X) \leq 0$ $\quad j = 1,2,3, \dots, m$

$h_k(X) = 0$ $\quad k = 1,2,3, \dots, p$

The Kuhn-Tucker conditions can be written as

$$\frac{\partial f(X)}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(X)}{\partial x_i} + \sum_{k=1}^{p} \beta_k \frac{\partial h_k(X)}{\partial x_i} = 0 \qquad i = 1,2,3, \dots, n$$

$$\lambda_j g_j = 0 \qquad j = 1,2,3, \dots, m$$

$$g_j \leq 0 \qquad j = 1,2,3, \dots, m$$

$$h_k = 0 \qquad k = 1,2,3, \dots, p$$

$$\lambda_j \geq 0 \qquad j = 1,2,3, \dots, m$$

Constrained Optimization

Minimize $f(X) = x_1^2 + 2x_2^2 + 3x_3^2$

Subject to $g_1(X) = x_1 - x_2 - 2x_3 \leq 12$

$g_2(X) = x_1 + 2x_2 - 3x_3 \leq 8$