## MA15010H: Multi-variable Calculus

(Assignment 1: Limits and continuity) July - November, 2025

- 1. Let  $x, y \in \mathbb{R}^m$ . Show that ||x + y|| = ||x|| + ||y|| iff y = 0 or  $x = \alpha y$  for some  $\alpha \ge 0$ .
- 2. Let  $x, y \in \mathbb{R}^m$  and r, s > 0. Show that  $B_r[x] \cap B_s[y] \neq \emptyset$  iff  $||x y|| \leq r + s$ .
- 3. Let  $(x_n)$  be a sequence in  $\mathbb{R}^m$ . Show that  $(x_n)$  converges in  $\mathbb{R}^m$  iff for each  $x \in \mathbb{R}^m$ , the sequence  $(x_n \cdot x)$  converges in  $\mathbb{R}$ .
- 4. State TRUE or FALSE with justification for each statement:
  - (a) If  $(x_n)$  in  $\mathbb{R}^m$  has no convergent subsequence, then it is necessary that  $\lim_{n\to\infty} ||x_n|| = \infty$ .
  - (b) If  $(x_n, y_n)$  is a bounded sequence in  $\mathbb{R}^2$  and every convergent subsequence of  $(x_n, y_n)$  converges to (0, 1), then  $(x_n, y_n)$  must converge to (0, 1).
- 5. Let  $f: \mathbb{R}^2 \to \mathbb{R}$  be defined by  $f(x,y) = \begin{cases} \frac{xy}{x^2 y^2} & \text{if } x^2 \neq y^2, \\ 0 & \text{if } x^2 = y^2. \end{cases}$

Determine all points in  $\mathbb{R}^2$  where f is continuous.

- 6. Let  $\alpha, \beta > 0$  and define  $f : \mathbb{R}^2 \to \mathbb{R}$  by  $f(x,y) = \begin{cases} \frac{|x|^{\alpha}|y|^{\beta}}{\sqrt{x^2 + y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$  Show that f is continuous iff  $\alpha + \beta > 1$ .
- 7. Let  $f: S \subseteq \mathbb{R}^2 \to \mathbb{R}$ , and  $(x_0, y_0) \in S$ . Let  $A = \{x \in \mathbb{R} : (x, y_0) \in S\}$ , and  $B = \{y \in \mathbb{R} : (x_0, y) \in S\}$ . Define  $\varphi(x) = f(x, y_0)$  for  $x \in A$ , and  $\psi(y) = f(x_0, y)$  for  $y \in B$ . If f is continuous at  $(x_0, y_0)$ , show  $\varphi: A \to \mathbb{R}$  is continuous at  $x_0$ , and  $\psi: B \to \mathbb{R}$  is continuous at  $y_0$ . Is the converse true? Justify.
- 8. If  $S = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 3\}$ , determine (with justification)  $S^0$ .