MA224: Real Analysis

Assignment 1: Metric Spaces and Normed Linear Spaces

January–April 2026

1. For each statement below, determine whether it is **true** or **false**. Provide a brief justification.
   (a) There exists a metric space having exactly 36 open sets.
   (b) It is impossible to define a metric d on $\mathbb{R}$ such that only finitely many subsets of $\mathbb{R}$ are open in $(\mathbb{R}, d)$.
   (c) If A and B are open (closed) subsets of a normed vector space X, then $A + B = \{a + b : a \in A, b \in B\}$ is open (closed) in X.
   (d) If A and B are closed subsets of $[0, \infty)$ (with the usual metric), then $A + B$ is closed in $[0, \infty)$.
   (e) It is possible to define a metric d on $\mathbb{R}$ such that the sequence $(1, 0, 1, 0, ...)$ converges in $(\mathbb{R}, d)$.
   (f) It is possible to define a metric d on $\mathbb{R}^2$ such that $((\frac{1}{n}, \frac{n}{n+1}))$ is not a Cauchy sequence in $(\mathbb{R}^2, d)$.
   (g) It is possible to define a metric d on $\mathbb{R}^2$ such that in $(\mathbb{R}^2, d)$, the sequence $((\frac{1}{n}, 0))$ converges but the sequence $((\frac{1}{n}, \frac{1}{n}))$ does not converge.
   (h) There exist two non-empty disjoint sets A and B in $\mathbb{R}$ such that $\inf\{|x - y| : x \in A \text{ and } y \in B\} = 0$.
   (i) If (x_n) is a sequence in a complete normed vector space X such that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, then (x_n) must converge in X.
   (j) If (f_n) is a sequence in $C[0, 1]$ such that $|f_{n+1}(x) - f_n(x)| \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and for all $x \in [0, 1]$, then there must exist $f \in C[0, 1]$ such that $\int_0^1 |f_n(x) - f(x)|\, dx \to 0$ as $n \to \infty$.
   (k) If (x_n) is a Cauchy sequence in a normed vector space, then $\lim_{n\to\infty} \|x_n\|$ must exist.
   (l) $\{f \in C[0, 1] : \|f\|_1 \leq 1\}$ is a bounded subset of the normed vector space $(C[0, 1], \|\cdot\|_\infty)$.
   (m) There exists a set $A \subset (\mathbb{R}, u)$ such that $\delta\left(A^o \cup \{0\}\right) = 0$ but $\delta\left((\bar{A})^o\right) = 1$, where δ stands for diameter.
   (n) For $x, y \in \ell^\infty$, $d(x, y) = \min\{1, \limsup_{n\to\infty} |x_n - y_n|\}$ defines a metric on ℓ^∞.
   (o) The sequence $f_n(t) = e^{-n^2 \sin \pi t}$ converges uniformly to 0 on $(0, 1)$.

2. For each of the following choices of X and d, determine whether (X, d) is a metric space.
   (a) $X = \mathbb{R}$ and $d(x, y) = \frac{|x-y|}{1+|xy|}$ for all $x, y \in \mathbb{R}$.
   (b) $X = \mathbb{R}$ and $d(x, y) = \min\{\sqrt{|x - y|}, |x - y|^2\}$ for all $x, y \in \mathbb{R}$.
   (c) $X = \mathbb{R}$ and $d(x, y) = |x - y|^p$ for all $x, y \in \mathbb{R}$ $(0 < p < 1)$.
   (d) $X = \mathbb{R}$ and for all $x, y \in \mathbb{R}$, $d(x, y) = \begin{cases} 1 + |x - y| & \text{if exactly one of } x \text{ and } y \text{ is positive,} \\ |x - y| & \text{otherwise.} \end{cases}$
   (e) $X = \mathbb{R}^2$ and $d(x, y) = (|x_1 - y_1| + |x_2 - y_2|^{\frac{1}{2}})^{\frac{1}{2}}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.
   (f) $X = \mathbb{R}^n$ and $d(x, y) = [(x_1 - y_1)^2 + \frac{1}{2}(x_2 - y_2)^2 + \cdots + \frac{1}{n}(x_n - y_n)^2]^{\frac{1}{2}}$ for all $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$.
   (g) $X = \mathbb{C}$ and for all $z, w \in \mathbb{C}$, $d(z, w) = \begin{cases} \min\{|z| + |w|, |z - 1| + |w - 1|\} & \text{if } z \neq w, \\ 0 & \text{if } z = w. \end{cases}$
   (h) $X = \mathbb{C}$ and for all $z, w \in \mathbb{C}$, $d(z, w) = \begin{cases} |z - w| & \text{if } \frac{z}{|z|} = \frac{w}{|w|}, \\ |z| + |w| & \text{otherwise.} \end{cases}$
   (i) $X = \mathbb{C}$ and $d(z, w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for all $z, w \in \mathbb{C}$.
   (j) $X =$ The class of all finite subsets of a nonempty set and $d(A, B) =$ The number of elements of the set $A \triangle B$ (the symmetric difference of A and B).

(k) $X = C[0,1]$ and $d(f,g) = (\int_0^1 |f(t) - g(t)|^2\, dt)^{\frac{1}{2}}$ for all $f, g \in C[0,1]$.

3. For each of the following definitions, determine whether $\|\cdot\|$ is a norm on $\mathbb{R}^2$, where for each $(x,y) \in \mathbb{R}^2$,
(a) $\|(x,y)\| = (\sqrt{|x|} + \sqrt{|y|})^2$.
(b) $\|(x,y)\| = \sqrt{\frac{x^2}{9} + \frac{y^2}{4}}$.
(c) $\|(x,y)\| = \begin{cases} \sqrt{x^2+y^2} & \text{if } xy \geq 0, \\ \max\{|x|, |y|\} & \text{if } xy < 0. \end{cases}$

4. If $\mathbf{x} \in \mathbb{R}^n$, then show that $\lim\limits_{p\to\infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty$.

5. If $1 \leq p < q \leq \infty$, then show that $\|x\|_q \leq \|x\|_p$ for all $x \in \ell^p$.

6. Determine whether $\|\cdot\|$ is a norm on $C[0,1]$, where for each $f \in C[0,1]$,
(a) $\|f\| = \min\{\|f\|_\infty, 2\|f\|_1\}$. (b) $\|f\| = \sup\{t\,|f(t)| : t \in [0,1]\}$.

7. Show that
(a) if $\mathbf{x} \in \mathbb{R}^n$, then $\lim\limits_{p\to\infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty$.
(b) if $x \in \ell^q$ for some $1 \leq q < \infty$, then $\lim\limits_{p\to\infty} \|x\|_p = \|x\|_\infty$.
(c) if $f \in C[a,b]$, then $\lim\limits_{p\to\infty} \|f\|_p = \|f\|_\infty$.

8. Let d be a metric on a real vector space X satisfying the following two conditions:
(i) $d(x+z, y+z) = d(x,y)$ for all $x, y, z \in X$,
(ii) $d(\alpha x, \alpha y) = |\alpha| d(x,y)$ for all $x, y \in X$ and for all $\alpha \in \mathbb{R}$.
Show that there exists a norm $\|\cdot\|$ on X such that $d(x,y) = \|x - y\|$ for all $x, y \in X$.

9. Let $\mathbb{R}^\infty$ be the real vector space of all sequences in $\mathbb{R}$, where addition and scalar multiplication are defined componentwise. Let $d((x_n),(y_n)) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1+|x_n - y_n|}$ for all $(x_n), (y_n) \in \mathbb{R}^\infty$. Show that d is a metric on $\mathbb{R}^\infty$ but that no norm on $\mathbb{R}^\infty$ induces d.

10. Let $(X, \|\cdot\|)$ be a nonzero normed vector space. Consider the metrics d_1, d_2 and d_3 on X:

$$d_1(x,y) := \min\{1, \|x-y\|\},$$

$$d_2(x,y) := \frac{\|x-y\|}{1+\|x-y\|},$$

$$d_3(x,y) := \begin{cases} 1 + \|x-y\| & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

for all $x, y \in X$. Prove that none of d_1, d_2 and d_3 is induced by any norm on X.

11. Let X be a normed vector space containing more than one point, let $x, y \in X$ and let $\varepsilon, \delta > 0$. If $B_\varepsilon[x] = B_\delta[y]$, show that $x = y$ and $\varepsilon = \delta$. Does the result remain true if X is assumed to be a metric space? Justify.

12. Determine whether the following sets are open and/or closed in $\mathbb{R}^2$ (with the usual metric).
(a) $\{(x,y) \in \mathbb{R}^2 : xy > 0\}$
(b) $\{(x,x) : x \in \mathbb{R}\}$
(c) $(0,1) \times \{0\}$
(d) $\{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$

(e) $\{(x, y) \in \mathbb{R}^2 : x + y < 1\}$
(f) $\{(x, y) \in \mathbb{R}^2 : y \in \mathbb{Z}\}$

13. Let $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Examine whether $A \cap B$ is a closed/an open subset of $\mathbb{R}^3$ with respect to the usual metric on $\mathbb{R}^3$.

14. Determine whether a finite subset of a metric space is open and/or closed.

15. For all $x, y \in \mathbb{R}$, let $d_1(x, y) = |x - y|$, $d_2(x, y) = \min\{1, |x - y|\}$ and $d_3(x, y) = \frac{|x-y|}{1+|x-y|}$. If G is an open set in any one of the three metric spaces $(\mathbb{R}, d_i)$ $(i = 1, 2, 3)$, then show that G is also open in the other two metric spaces.

16. Let X be a nonzero normed vector space. Show that $\{x \in X : \|x\| < 1\}$ is not closed in X and $\{x \in X : \|x\| \leq 1\}$ is not open in X.

17. Show that $A = \{f \in C[0, 1] : \|f\|_1 < 1\}$ is an unbounded subset of the normed linear space $(C[0, 1], \|\cdot\|_\infty)$.

18. Let X be a normed vector space and let $Y (\neq X)$ be a subspace of X. Show that Y is not open in X.

19. Let F_n be a sequence of closed sets in $\mathbb{R}$ such that $F_n \subset (n, n + 1]$ and $F_n \cap F_m = \emptyset$, whenever $m \neq n$. Show that $F = \bigcup\limits_{n=1}^{\infty} F_n$ is a closed set in $\mathbb{R}$.

20. Let $A(\neq \emptyset) \subset \mathbb{R}^n$ be such that every continuous function $f : A \to \mathbb{R}$ is bounded. Show that A is a compact subset of $\mathbb{R}^n$.

21. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous such that $\lim\limits_{\|\mathbf{x}\|_2 \to \infty} f(\mathbf{x}) = 1$. Show that f is bounded on $\mathbb{R}^2$.

22. Let X, Y be metric spaces and let X be compact. If $f : X \to Y$ is a bijective continuous function, then show that $f^{-1} : Y \to X$ is continuous.

23. Let $\alpha > 0$ and let $f : (\mathbb{R}^n, \|\cdot\|_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ be continuous such that $\|f(\mathbf{x}) - f(\mathbf{y})\|_2 \geq \alpha\|\mathbf{x} - \mathbf{y}\|_2$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that the range of f is a closed subset of $(\mathbb{R}^m, \|\cdot\|_2)$.

24. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Show that $\overline{f(A)} \subset f(\overline{A})$ for all $A \subset [0, 1]$.

25. Let $f : X \to Y$ be continuous, where X and Y are metric spaces. If $x \in X$ is a limit point of $A \subset X$, then show that $f(x) \in f(A)$ or $f(x)$ is a limit point of $f(A)$.

26. Let $f : (X, d) \to \mathbb{R}$ be a continuous function. Show that $\{x \in X : f(x) \neq 0\}$ is an open set in the metric space (X, d).

27. Let (x_n) and (y_n) be Cauchy sequences in a metric space (X, d). Show that the sequence $(d(x_n, y_n))$ is convergent.

28. Let (x_n) be a sequence in a complete metric space (X, d) such that $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. Show that (x_n) converges in (X, d).

29. Let (x_n) be a sequence in a metric space X such that each of the subsequences (x_{2n}), (x_{2n-1}) and (x_{3n}) converges in X. Show that (x_n) converges in X.

30. Show that each of the following metric spaces is not complete.
(a) $(\mathbb{N}, d)$, where $d(m,n) = |\frac{1}{m} - \frac{1}{n}|$ for all $m, n \in \mathbb{N}$
(b) $((0,\infty), d)$, where $d(x,y) = |\frac{1}{x} - \frac{1}{y}|$ for all $x, y \in (0,\infty)$
(c) $(\mathbb{R}, d)$, where $d(x,y) = |\frac{x}{1+|x|} - \frac{y}{1+|y|}|$ for all $x, y \in \mathbb{R}$
(d) $(\mathbb{R}, d)$, where $d(x,y) = |e^x - e^y|$ for all $x, y \in \mathbb{R}$

31. Examine whether the following metric spaces are complete.
(a) $([0,1), d)$, where $d(x,y) = |\frac{x}{1-x} - \frac{y}{1-y}|$ for all $x, y \in [0,1)$
(b) $((-1,1), d)$, where $d(x,y) = |\tan\frac{\pi x}{2} - \tan\frac{\pi y}{2}|$ for all $x, y \in (-1,1)$

32. For $X(\neq \emptyset) \subset \mathbb{R}$, let $d(x,y) = \frac{|x-y|}{1+|x-y|}$ for all $x, y \in X$. Determine whether the metric space (X, d) is complete, where X is
(a) $[0,1] \cap \mathbb{Q}$.
(b) $[-1,0] \cup [1,\infty)$.
(c) $\{n^2 : n \in \mathbb{N}\}$.

33. For $f \in C^1[0,1]$, define $\|f\| = \|f\|_1 + \|f\|_\infty$. Determine whether $(C^1[0,1], \|\cdot\|)$ is a complete normed linear space.

34. Determine whether the sequence (f_n) converges in $(C[0,1], d_\infty)$, where for all $n \in \mathbb{N}$ and for all $t \in [0,1]$,
(a) $f_n(t) = \frac{nt^2}{1+nt}$.
(b) $f_n(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}$.
(c) $f_n(t) = \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n}, \\ \frac{1}{nt} & \text{if } \frac{1}{n} < t \le 1. \end{cases}$
(d) $f_n(t) = \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n}, \\ \frac{n}{n-1}(1-t) & \text{if } \frac{1}{n} < t \le 1. \end{cases}$

35. Let $f : [0,1] \to \mathbb{R}$ be continuous and $f(1) = 0$. If $f_n(x) = f(x)x^n$ for all $x \in [0,1]$ and for all $n \in \mathbb{N}$, then examine whether the sequence (f_n) converges uniformly on $[0,1]$.

36. Let X be a metric space and let $f, g, f_n, g_n : X \to \mathbb{R}$ $(n \in \mathbb{N})$ be such that $f_n \to f$ uniformly on X and $g_n \to g$ uniformly on X.
(a) Show that $f_n + g_n \to f + g$ uniformly on X.
(b) Is it necessary that $f_n.g_n \to f.g$ uniformly on X? Justify.
(c) If f and g are bounded on X, then show that $f_n.g_n \to f.g$ uniformly on X.

37. Let X be a metric space and let (f_n) be a sequence of real-valued bounded functions on X. If $f : X \to \mathbb{R}$ is such that $f_n \to f$ uniformly on X, then show that f is bounded on X.
Does this result hold if $f_n \to f$ pointwise on X? Justify.

38. Let (f_n) be a uniformly convergent sequence of real-valued bounded functions defined on a metric space X. Show that there exists $M > 0$ such that $|f_n(x)| \le M$ for all $x \in X$ and for all

$n \in \mathbb{N}$.

39. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} x^{2n-1}$ is uniformly convergent in $\mathbb{R}$.

40. Let X be a normed vector space and let $x \in X$. Show that $\|x\| = \inf\left\{\frac{1}{|\alpha|} : \alpha \in \mathbb{R} \setminus \{0\},\ \|\alpha x\| \leq 1\right\}$.

41. Let X be a normed vector space and let $x, y \in X$. Show that $\|x\| \leq \max\{\|x+y\|, \|x-y\|\}$.

42. Let X be a vector space over $\mathbb{R}$ and let $p : X \to [0, \infty)$ satisfy the following three conditions:
    (i) $p(x) = 0$ iff $x = 0$ $(x \in X)$.
    (ii) $p(\alpha x) = |\alpha| p(x)$ for all $x \in X$ and for all $\alpha \in \mathbb{R}$.
    (iii) $\{x \in X : p(x) \leq 1\}$ is a convex subset of X.
    Show that p is a norm on X.
    (A subset S of X is said to be convex if $(1-t)x + ty \in S$ for all $x, y \in S$ and for all $t \in [0, 1]$.)

43. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and let there exist $\alpha > 0$ such that $\|f(\mathbf{x}) - f(\mathbf{y})\| \geq \alpha \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that $f(\mathbb{R}^n)$ is complete.

44. Let X be a complete metric space. Let $f : X \to X$ be a contraction and let $g : X \to X$ be such that $f \circ g = g \circ f$. Show that g has a fixed point in X. Is it necessary that the fixed point of g in X is unique? Justify.

45. Let (X, d) be a complete metric space and $f : X \to X$ be such that $f^m : X \to X$ is a contraction for some $m \in \mathbb{N}$. Show that f has a unique fixed point in X.

46. Let (X, d) be a compact metric space and let $f : X \to X$ be such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Show that f has a fixed point in X.

47. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction and $g(\mathbf{x}) = \mathbf{x} - f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Show that $g : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one and onto. Also, show that both g and $g^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

48. Using the contraction mapping theorem, show that the equation $4x^5 - 2x^2 - 4x + 1 = 0$ has exactly one root in $\left(0, \frac{1}{2}\right)$.

49. Using the contraction mapping theorem, show that the initial value problem $\frac{dy}{dx} = x + \sin(x^2 y)$ for $x \in [0, 1]$, $y(0) = \frac{1}{2}$, has a unique solution.

50. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction and let $g(\mathbf{x}) = \mathbf{x} - f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Show that $g : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one and onto. Also, show that both g and $g^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

51. Let $a_1 = 1$ and $a_{n+1} = \sqrt[3]{1 + a_n^2}$. Using fixed point theory, prove that the sequence $\{a_n\}$ is convergent, and that its limit satisfies the equation $x^3 - x^2 - 1 = 0$.

52. Using fixed point theory, determine all functions $f \in C[0, 1]$ such that $f(x) = \int_0^x (x - y) f(y) dy$.

53. Let $B = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 < 1\}$ and let $f : (B, d_2) \to (B, d_2)$ be continuous such that $\|f(\mathbf{x})\|_2 < \|\mathbf{x}\|_2$ for all $\mathbf{x} \in B \setminus \{\mathbf{0}\}$. Let $\mathbf{x}_1 \in B \setminus \{\mathbf{0}\}$ and let $\mathbf{x}_{k+1} = f(\mathbf{x}_k)$ for all $k \in \mathbb{N}$. Show that $\mathbf{x}_k \to \mathbf{0}$.