MA642: Real Analysis -1

(Assignment 1: Metric and Normed Linear Spaces) July - November, 2025

- 1. State TRUE or FALSE giving proper justification for each of the following statements.
 - (a) There does not exist a monotone function $f: \mathbb{R} \to \mathbb{Q}$ which is onto.
 - (b) There exists a monotone function $f:(0,\infty)\to\mathbb{R}$ such that each $c\in(0,\infty)$ satisfies $|f(c+)-f(c-)|=\frac{1}{c}$.
 - (c) There exists a sequence of differentiable functions f_n on $(0, \infty)$ such that f'_n is uniformly convergent on $(0, \infty)$ but f_n is nowhere point-wise convergent.
 - (d) There exists a metric space having exactly 36 open sets.
 - (e) It is impossible to define a metric d on \mathbb{R} such that only finitely many subsets of \mathbb{R} are open in (\mathbb{R}, d) .
 - (f) If A and B are open (closed) subsets of a normed vector space X, then $A + B = \{a + b : a \in A, b \in B\}$ is open (closed) in X.
 - (g) If A and B are closed subsets of $[0, \infty)$ (with the usual metric), then A + B is closed in $[0, \infty)$.
 - (h) It is possible to define a metric d on \mathbb{R} such that the sequence (1,0,1,0,...) converges in (\mathbb{R},d) .
 - (i) It is possible to define a metric d on \mathbb{R}^2 such that $((\frac{1}{n}, \frac{n}{n+1}))$ is not a Cauchy sequence in (\mathbb{R}^2, d) .
 - (j) It is possible to define a metric d on \mathbb{R}^2 such that in (\mathbb{R}^2, d) , the sequence $((\frac{1}{n}, 0))$ converges but the sequence $((\frac{1}{n}, \frac{1}{n}))$ does not converge.
 - (k) Let $A \subset (1, \infty)$ be a closed set. Then $A^2 := \{a^2 : a \in A\}$ is a closed set.
 - (1) Let $A_n = \{(x,y) \in \mathbb{R}^2 : 0 < \frac{1}{x} < y < \frac{1}{n}\}$. Whether the set $\bigcap_{n=1}^{\infty} A_n$ is open/closed?
 - (m) There exist a set $A \subset (\mathbb{R}, u)$ such that $\delta(A^o \cup \{0\}) = 0$ but $\delta((\bar{A})^o) = 1$, where δ stands for diameter.
 - (n) If (x_n) is a sequence in a complete normed vector space X such that $||x_{n+1} x_n|| \to 0$ as $n \to \infty$, then (x_n) must converge in X.
 - (o) If (f_n) is a sequence in C[0,1] such that $|f_{n+1}(x) f_n(x)| \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and for all $x \in [0,1]$, then there must exist $f \in C[0,1]$ such that $\int_0^1 |f_n(x) f(x)| dx \to 0$ as $n \to \infty$.
 - (p) If (x_n) is a Cauchy sequence in a normed vector space, then $\lim_{n\to\infty} ||x_n||$ must exist.
 - (q) $\{f \in C[0,1] : ||f||_1 \le 1\}$ is a bounded subset of the normed vector space $(C[0,1], ||\cdot||_{\infty})$.
 - (r) For $x, y \in l^{\infty}$, $d(x, y) = \min\{1, \limsup |x_n y_n|\}$ define a metric on l^{∞} .
 - (s) The sequence $f_n(t) = e^{-n^2 \sin \pi t}$ converge uniformly to 0 on (0,1).
 - (t) If sequence (x_n) in \mathbb{R} satisfies $0 \leq \inf x_n \sup x_n < \infty$, then x_n has necessarily convergent subsequence.
 - (u) If $f: \mathbb{R} \to \mathbb{R}$ is continuous, bounded and monotone function, then $\lim_{x \to \pm \infty} f(x)$ are necessarily finite?
- 2. What is the cardinality of the set $\{f: \mathbb{R} \to \mathbb{R}, f \text{ is nowhere continuous}\}$?
- 3. For a monotone increasing function $f:[a,b]\to\mathbb{R}$, define $g(x)=\sup\{f(y):y< x\}$. If f has limit at c, then show that f(c)=g(c).
- 4. Examine whether (X, d) is a metric space, where
 - (a) $X = \mathbb{R}$ and $d(x, y) = \frac{|x-y|}{1+|xy|}$ for all $x, y \in \mathbb{R}$.
 - (b) $X = \mathbb{R}$ and $d(x, y) = |x y|^p$ for all $x, y \in \mathbb{R}$ (0 .
 - (c) $X = \mathbb{R}$ and $d(x, y) = \min\{\sqrt{|x y|}, |x y|^2\}$ for all $x, y \in \mathbb{R}$.

- (d) $X = \mathbb{R}$ and for all $x, y \in \mathbb{R}$, $d(x, y) = \begin{cases} 1 + |x y| & \text{if exactly one of } x \text{ and } y \text{ is positive,} \\ |x y| & \text{otherwise.} \end{cases}$
- (e) $X = \mathbb{R}^2$ and $d(x,y) = (|x_1 y_1| + |x_2 y_2|^{\frac{1}{2}})^{\frac{1}{2}}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. (f) $X = \mathbb{R}^n$ and $d(x,y) = [(x_1 y_1)^2 + \frac{1}{2}(x_2 y_2)^2 + \dots + \frac{1}{n}(x_n y_n)^2]^{\frac{1}{2}}$ for all $x = (x_1, \dots, x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n.$
- $(g) \ X = \mathbb{C} \text{ and for all } z, w \in \mathbb{C}, \ d(z, w) = \begin{cases} \min\{|z| + |w|, |z 1| + |w 1| & \text{if } z \neq w, \\ 0 & \text{if } z = w. \end{cases}$ $(h) \ X = \mathbb{C} \text{ and for all } z, w \in \mathbb{C}, \ d(z, w) = \begin{cases} |z w| & \text{if } \frac{z}{|z|} = \frac{w}{|w|}, \\ |z| + |w| & \text{otherwise.} \end{cases}$
- (i) $X = \mathbb{C}$ and $d(z, w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for all $z, w \in \mathbb{C}$.
- (j) X = The class of all finite subsets of a nonempty set and d(A, B) = The number of elementsof the set $A \triangle B$ (the symmetric difference of A and B).
- 5. Let $1 \le p \le \infty$ and d_i ; i = 1, 2 be two metric on a non-empty set X. Show that $d_p = (d_1^p + d_2^p)^{1/p}$ is a metric on X for $1 \leq p < \infty$. Whether $d_{\infty} = \max\{d_1, d_2\}$ is a metric on X?
- 6. Examine whether $\|\cdot\|$ is a norm on \mathbb{R}^2 , where for each $(x,y)\in\mathbb{R}^2$,
 - (a) $||(x,y)|| = (|x|^p + |y|^p)^{\frac{1}{p}}$, where 0 .

 - (b) $\|(x,y)\| = \sqrt{\frac{x^2}{9} + \frac{y^2}{4}}$. (c) $\|(x,y)\| = \begin{cases} \sqrt{x^2 + y^2} & \text{if } xy \ge 0, \\ \max\{|x|,|y|\} & \text{if } xy < 0. \end{cases}$
- 7. Let $||f|| = \min\{||f||_{\infty}, 2||f||_1\}$ for all $f \in C[0, 1]$. Prove that $||\cdot||$ is not a norm on C[0, 1].
- 8. Let X be a normed linear space. Prove that norm of any $x \in X$, can be expressed as ||x|| = $\inf \{ |\alpha| : \alpha \in \mathbb{C} \setminus \{0\} \text{ with } ||x|| \le |\alpha| \}.$
- 9. Let $(X, \|\cdot\|)$ be a normed linear space. Show that $\|x\| = \sup\{|\alpha| : |\alpha| < \|x\|\}$.
- 10. Let $(X, \|\cdot\|)$ be a normed linear space and let p be a seminorm on X. Show that $p: (X, \|\cdot\|) \to \mathbb{R}$ is continuous iff there exists $\alpha > 0$ such that $p(x) \leq \alpha ||x||$ for all $x \in X$.
- 11. If $1 \le p < q \le \infty$, then show that $||x||_q \le ||x||_p$ for all $x \in \ell^p$.
- 12. If $\mathbf{x} \in \mathbb{R}^n$, then show that $\lim_{p \to \infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_{\infty}$. And if $x \in l^p$, then show that $\liminf_{p \to \infty} \|\mathbf{x}\|_p \ge \|\mathbf{x}\|_{\infty}$.
- 13. Let d be a metric on a real vector space X satisfying the following two conditions:
 - (i) d(x+z, y+z) = d(x, y) for all $x, y, z \in X$,
 - (ii) $d(\alpha x, \alpha y) = |\alpha| d(x, y)$ for all $x, y \in X$ and for all $\alpha \in \mathbb{R}$.

Show that there exists a norm $\|\cdot\|$ on X such that $d(x,y) = \|x-y\|$ for all $x,y \in X$.

- 14. Let f be a non-negative function on a linear space X such that $f(\alpha x) = |\alpha| f(x)$ for all $\alpha \in \mathbb{C}$. Show that f is norm on X if and only if f is a convex map which can vanish at most at one point.
- 15. Let $f:(X,d)\to [0,1]$ be continuous map. Show that $f^{-1}(0)$ is a closed G_{δ} set.
- 16. Let (x_n) be a sequence in a normed linear space X which converges to a non-zero vector $x \in X$. Show that $\frac{x_1+\cdots+x_n}{n^{\alpha}} \to x$ if and only if $\alpha = 1$. What are admissible values of α if $x_n \to 0$?
- 17. For $x = (x_n) \in l^2$, write $||x|| = (\sum_{n=1}^{\infty} a_n |x_n|^2)^{1/2}$. Find all possible sequences (a_n) such that $||\cdot||$ is a norm on l^2 .
- 18. Let \mathbb{R}^{∞} be the real vector space of all sequences in \mathbb{R} , where addition and scalar multiplication are defined componentwise. Let $d((x_n), (y_n)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|}$ for all $(x_n), (y_n) \in \mathbb{R}^{\infty}$. Show that d is a metric on \mathbb{R}^{∞} but that no norm on \mathbb{R}^{∞} induces

19. Let $(X, \|\cdot\|)$ be a nonzero normed vector space. Consider the metrics d_1, d_2 and d_3 on X:

$$d_1(x,y) := \min\{1, ||x - y||\},$$

$$d_2(x,y) := \frac{||x - y||}{1 + ||x - y||},$$

$$d_3(x,y) := \begin{cases} 1 + ||x - y|| & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

for all $x, y \in X$. Prove that none of d_1, d_2 and d_3 is induced by any norm on X.

- 20. Let X be a normed vector space containing more than one point, let $x, y \in X$ and let $\varepsilon, \delta > 0$. If $B_{\varepsilon}[x] = B_{\delta}[y]$, show that x = y and $\varepsilon = \delta$. Does the result remain true if X is assumed to be a metric space? Justify.
- 21. Let $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Examine whether $A \cap B$ is a closed/an open subset of \mathbb{R}^3 with respect to the usual metric on \mathbb{R}^3 .
- 22. Let F_n be a sequence of closed sets in \mathbb{R} such that $F_n \subset (n, n+1]$ and $F_n \cap F_m = \emptyset$, whenever $m \neq n$. Show that $F = \bigcup_{n=1}^{\infty} F_n$ is a closed set in \mathbb{R} .
- 23. For all $x, y \in \mathbb{R}$, let $d_1(x, y) = |x y|$, $d_2(x, y) = \min\{1, |x y|\}$ and $d_3(x, y) = \frac{|x y|}{1 + |x y|}$. If G is an open set in any one of the three metric spaces (\mathbb{R}, d_i) (i = 1, 2, 3), then show that G is also open in the other two metric spaces.
- 24. Let X be a normed vector space and let $Y(\neq X)$ be a subspace of X. Show that Y is not open
- 25. Let (x_n) and (y_n) be Cauchy sequences in a metric space (X,d). Show that the sequence $(d(x_n,y_n))$ is convergent.
- 26. Let d_o be the discrete metric on non-empty set X. Show that (X, d_o) is complete.
- 27. Let (x_n) be a sequence in a complete metric space (X,d) such that $\sum_{n=1}^{\infty} d(x_n,x_{n+1}) < \infty$. Show that (x_n) converges in (X, d).
- 28. Let (x_n) be a sequence in a metric space X such that each of the subsequences (x_{2n}) , (x_{2n-1}) and (x_{3n}) converges in X. Show that (x_n) converges in X.
- 29. Show that the following are incomplete metric spaces.

 - (a) (\mathbb{N}, d) , where $d(m, n) = \left|\frac{1}{m} \frac{1}{n}\right|$ for all $m, n \in \mathbb{N}$ (b) $((0, \infty), d)$, where $d(x, y) = \left|\frac{1}{x} \frac{1}{y}\right|$ for all $x, y \in (0, \infty)$ (c) (\mathbb{R}, d) , where $d(x, y) = \left|\frac{x}{1+|x|} \frac{y}{1+|y|}\right|$ for all $x, y \in \mathbb{R}$ (d) (\mathbb{R}, d) , where $d(x, y) = |e^x e^y|$ for all $x, y \in \mathbb{R}$
- 30. Examine whether the following metric spaces are complete.

 - (a) ([0,1), d), where $d(x,y) = \left| \frac{x}{1-x} \frac{y}{1-y} \right|$ for all $x, y \in [0,1)$ (b) ((-1,1), d), where $d(x,y) = \left| \tan \frac{\pi x}{2} \tan \frac{\pi y}{2} \right|$ for all $x, y \in (-1,1)$
 - (c) ((0,2],d), where $d(x,y) = |\frac{1}{x} \frac{1}{y}|$
- 31. For $X(\neq \emptyset) \subset \mathbb{R}$, let $d(x,y) = \frac{|x-y|}{1+|x-y|}$ for all $x,y \in X$. Examine the completeness of the metric space (X, d), where X is
 - (a) $[0,1] \cap \mathbb{Q}$.
 - (b) $[-1,0] \cup [1,\infty)$.
 - (c) $\{n^2 : n \in \mathbb{N}\}.$

- 32. Let X = C[0,1] be the space all the continuous functions on interval [0,1]. Prove that norms $\| \cdot \|_{\infty}$ and $\| \cdot \|_{1}$ on X are not equivalent.
- 33. Let $C^1[0,1]$ denote the space of all continuously differentiable functions on [0,1]. For $f \in$ $C^{1}[0,1]$, define $||f|| = ||f||_{\infty} + ||f'||_{\infty}$. Show that space $(C^{1}[0,1], ||.||)$ is a Banach space.
- 34. The space $(C^1[0,1], \|.\|)$, where $\|f\| = (\|f\|_2^2 + \|f'\|_2^2)^{\frac{1}{2}}$ is complete.
- 35. Let $f \in C^1[0,1]$ and $||f|| = ||f'||_2 + ||f||_{\infty}$. Then verify if $(C^1[0,1], ||.||)$ is complete.
- 36. Let $f \in C^1[0,1]$. Then verify if $||f|| = \min(||f'||_2, ||f||_{\infty})$ defines a norm on $C^1[0,1]$.
- 37. Let $X = \{f \in C^1[0,1] : f(0) = 0\}$. Then $||f|| = ||f'||_2$ is a norm on $C^1[0,1]$ but not complete.
- 38. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Let X be the class of all functions f which are analytic on D and continuous on \bar{D} . Define $||f|| = \sup\{|f(e^{it})| : 0 \le t \le 2\pi\}$. Show that (X, ||.||) is complete.
- 39. Examine whether the sequence (f_n) is convergent in $(C[0,1],d_\infty)$, where for all $n\in\mathbb{N}$ and for all $t \in [0, 1]$,
 - (a) $f_n(t) = \frac{nt^2}{1+nt}$.

 - (b) $f_n(t) = 1 + t + \frac{t^2}{2!} + \dots + \frac{t^n}{n!}$. (c) $f_n(t) = \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n}, \\ \frac{1}{nt} & \text{if } \frac{1}{n} < t \le 1. \end{cases}$
 - (d) $f_n(t) = \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n}, \\ \frac{n}{n-1}(1-t) & \text{if } \frac{1}{n} < t \le 1. \end{cases}$
- 40. Find the point-wise limit of the sequence $f_n(t) = e^{-nt^2} \sin nt$. Examine for uniform convergence of f_n on \mathbb{R} .
- 41. Let $f_n, f: \mathbb{R} \to (0, \infty)$ be such that $f_n \to f$ uniformly on \mathbb{R} . Examine for $e^{f_n} \to e^f$ uniformly
- 42. Let $f_n(t) = \sqrt{t^2 + n}$. Examine for the uniform convergence of f'_n on \mathbb{R} .
- 43. Let X be the class of all continuous functions $f: \mathbb{R} \to \mathbb{C}$ such that for each $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}$ such that $|f(x)| < \epsilon$, for all $x \in \mathbb{R} \setminus K$. Show that $(X, \| . \|_{\infty})$ is complete.
- 44. Let $1 \le p < \infty$. Let X_p be a class of all the Riemann integrable functions on [0, 1]. Prove that $||f||_p = \left(\int_0^1 |f|^p\right)^{\frac{1}{p}} < \infty$. Prove that $(X_p, || . ||_p)$ is a normed linear space but not complete. 45. Show that $\{(x_n) \in l^2 : |x_n| < \frac{1}{n} \text{ for all } n \in \mathbb{N}\}$ is a convex set with empty interior.
- 46. Suppose that $x \in l^p$ for some $p \geq 1$. Show that $\lim_{p \to \infty} \inf ||x||_p \geq ||x||_{\infty}$. Prove/disprove that $\lim ||x||_p = ||x||_{\infty}.$
- 47. Let M be a subspace of a normed linear space X. Then show that M is closed if and only if $\{y \in M : ||y|| \le 1\}$ is closed in X.
- 48. Let $T: (C[0,\frac{\pi}{2}],\|.\|_{\infty}) \to (C[0,\frac{\pi}{2}],\|.\|_{\infty})$ be defined by $(Tf)(x) = \int_{s=0}^{x} f(s) \sin s ds$. Show that T is not a contraction but T^2 is a contraction.
- 49. Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be continuous and let there exist $\alpha > 0$ such that $||f(\mathbf{x}) f(\mathbf{y})|| \ge \alpha ||\mathbf{x} \mathbf{y}||$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that $f(\mathbb{R}^n)$ is complete.
- 50. Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a contraction and let $g(\mathbf{x}) = \mathbf{x} f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Show that $g:\mathbb{R}^n\to\mathbb{R}^n$ is one-one and onto. Also, show that both g and $g^{-1}:\mathbb{R}^n\to\mathbb{R}^n$ are continuous.
- 51. Find a neighborhood of x=0 in which initial value problem $y'=\frac{x}{1+y^2}$ with y(0)=0 has a unique solution.