MA746: Fourier Analysis

(Assignment 1: Fourier Series)

July - November, 2025

1. Determine whether each of the following statements is **TRUE** or **FALSE**, providing rigorous justification in each case.
   (a) Let D_n denote the Dirichlet kernel on S^1. Does the identity $D_n * D_n = D_n$ necessarily hold?
   (b) Does there exist a function $f \in L^1(S^1)$ such that $\sum_{n=-\infty}^{\infty} |n\hat{f}(n)|^2 = \infty$?

2. Suppose f is continuously differentiable on S^1. Show that
$$\widehat{f'}(n) = in\hat{f}(n) \quad \text{for all } n \in \mathbb{Z}.$$
Deduce that there exists a constant $C > 0$ such that
$$|\hat{f}(n)| \leq \frac{C}{|n|}.$$
Does this conclusion remain valid if f is absolutely continuous?

3. Let f be of bounded variation on $[-\pi, \pi]$. Prove that
$$|\hat{f}(n)| \leq \frac{\mathrm{Var}(f)}{2\pi|n|}$$
for all $n \in \mathbb{Z}$.

4. For $f \in L^1(S^1)$, establish that
$$\hat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left[f(x) - f\left(x + \frac{\pi}{n}\right)\right]e^{-inx}\,dx.$$
Use this identity to prove the Riemann–Lebesgue lemma.

5. Let $f \in L^1(S^1)$ satisfy the Hölder condition
$$|f(x+h) - f(x)| \leq M|h|^{\alpha}$$
for all $x, h \in S^1$, where $0 < \alpha < 1$ and $M > 0$. Show that
$$\hat{f}(n) = O\left(\frac{1}{|n|^{\alpha}}\right).$$

6. Demonstrate that Fejér's kernel F_n can be expressed as
$$F_n(t) = \sum_{j=-n}^{n}\left(1 - \frac{|j|}{n}\right)e^{ijt}.$$

7. Given $f \in L^1(S^1)$ and $m \in \mathbb{N}$, define $f_m(t) = f(mt)$. Prove that
$$\hat{f}_m(n) = \begin{cases} \hat{f}\left(\frac{n}{m}\right), & \text{if } m \mid n, \\ 0, & \text{otherwise.} \end{cases}$$

8. For $f : S^1 \to \mathbb{C}$, and for $x, y \in S^1$, define the translation operator $\tau_x f(y) = f(x - y)$. Prove that the map $x \mapsto \tau_x f$ is continuous in $L^p(S^1)$ for $1 \leq p < \infty$. That is,
$$\|\tau_x f - f\|_p \to 0 \quad \text{as } |x| \to 0.$$
Does this continuity hold for $p = \infty$?

9. Let $f \in L^1(S^1)$ and $g \in L^{\infty}(S^1)$. Show that
$$\lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(nt)\,dt = \hat{f}(0)\hat{g}(0).$$

10. Given $f \in L^1(S^1)$, define the convolution operator $T_f : L^1(S^1) \to L^1(S^1)$ by $T_f(g) = f * g$. Prove that T_f is a bounded operator and that its operator norm satisfies

$$\|T_f\| = \|f\|_1.$$

11. Let P be a trigonometric polynomial of degree n on S^1. Show that

$$\|P'\|_\infty \leq 2n\|P\|_\infty.$$

12. For $1 \leq p \leq \infty$ with $p^{-1} + q^{-1} = 1$, and $f \in L^p(S^1)$, $g \in L^q(S^1)$, prove that the convolution $f * g$ is continuous on S^1.
13. Suppose $f \in L^\infty(S^1)$ satisfies

$$|\hat{f}(n)| \leq \frac{k}{|n|}$$

for some constant $k > 0$ and all $n \in \mathbb{Z} \setminus \{0\}$. Prove that

$$|S_n(f)(t)| \leq \|f\|_\infty + 2k,$$

where $S_n(f) = D_n * f$.
14. If f is a bounded monotone function on S^1, show that

$$\hat{f}(n) = O\left(\frac{1}{|n|}\right).$$

15. Let f be Riemann integrable on $[-\pi, \pi]$. Prove that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 < \infty,$$

from which it follows that $\hat{f}(n) = o(1)$.
16. Prove that if the series $\sum_{n=0}^{\infty} a_n$ of complex numbers converges to s, then it is both Cesàro and Abel summable to s.
17. Prove that if the series $\sum_{n=0}^{\infty} a_n$ is Cesàro summable to σ, then it is Abel summable to σ. Show by counterexample that the converse need not hold.
18. Suppose the series $\sum_{n=0}^{\infty} a_n$ is Cesàro summable to l. Show that

$$\lim_{n\to\infty} \frac{a_n}{n} = 0,$$

where $s_n = a_1 + \cdots + a_n$.
19. Define $u(r, \theta) = \frac{\partial P_r}{\partial \theta}(\theta)$, where $P_r(\theta)$ is the Poisson kernel on the open unit disk $\mathbb{D} = \{re^{i\theta} : 0 \leq r < 1, \theta \in [-\pi, \pi)\}$. Prove that

$$\Delta u = 0 \quad \text{on } \mathbb{D}$$

and

$$\lim_{r\to 1} u(r, \theta) = 0$$

for every $\theta \in [-\pi, \pi)$.
20. Let f be Riemann integrable on $[-\pi, \pi]$ and define the Abel mean

$$A_r(f)(\theta) = f * P_r(\theta), \quad 0 \leq r < 1.$$

If f has a jump discontinuity at θ, prove that

$$\lim_{r\to 1} A_r(f)(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2}.$$

Provide justification for why

$$\lim_{r\to 1} A_r(f)(\theta) \neq \frac{f(\theta)}{2}$$

when f is continuous at θ.

21. Let f be Riemann integrable on $[-\pi, \pi]$ and $\sigma_n(f)(\theta) = f * F_n(\theta)$, where F_n is Fejér's kernel. If f has a jump discontinuity at θ, prove that

$$\lim_{n\to\infty} \sigma_n(f)(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2}.$$

22. Suppose f is Riemann integrable on $[-\pi, \pi]$ such that

$$\hat{f}(n) = O\left(\frac{1}{|n|}\right) \quad \text{for all } n \in \mathbb{Z}.$$

Prove the following assertions:

(a) If f is continuous at θ, then

$$S_N(f)(\theta) = D_N * f(\theta) \to f(\theta) \quad \text{as } N \to \infty.$$

(b) If f has a jump discontinuity at θ, then

$$S_N(f)(\theta) \to \frac{f(\theta^+) + f(\theta^-)}{2} \quad \text{as } N \to \infty.$$

(c) If f is continuous on $[-\pi, \pi]$, then the convergence

$$S_N(f) \to f$$

is uniform.

23. Assume f is a Lebesgue measurable function on S^1 satisfying

$$\int_0^{2\pi} \frac{|f(t)|}{t}\, dt < \infty.$$

Show that

$$\lim_{n\to\infty} S_n(f; 0) = 0.$$

24. For $f \in L^2(S^1)$, prove that

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) \to \hat{f}(0)$$

in the L^2-metric as $n \to \infty$.

25. Does there exist a function $f \in L^1(S^1)$ such that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \infty?$$

26. Suppose $f \in L^1(S^1)$ vanishes on a neighborhood of $x = 0$. Prove that

$$S_N(f) \to 0$$

uniformly near $x = 0$.

27. Let f be a function on $[-\pi, \pi]$ satisfying the Lipschitz condition

$$|f(\theta) - f(\varphi)| \le M|\theta - \varphi|,$$

for some $M > 0$ and all $\theta, \varphi \in [-\pi, \pi]$.

(a) For

$$u(r, \theta) = P_r * f(\theta),$$

show that $\frac{\partial u}{\partial \theta}$ exists for all $0 \le r < 1$ and that

$$\left|\frac{\partial u}{\partial \theta}\right| \le M.$$

(b) Demonstrate that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| \le |\hat{f}(0)| + 2M\sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}}.$$

28. If f is continuously differentiable on S^1, show that

$$\sum_{n=-\infty}^{\infty} (1+|n|^2)|\hat{f}(n)|^2 < \infty.$$

29. Let $\{G_n\}_{n=1}^{\infty}$ be a family of good kernels on S^1. Prove that

$$\lim_{n\to\infty} \hat{G}_n(k) = 1.$$

30. Let f and g be Riemann integrable on $[-\pi, \pi]$. Define $\tilde{g}(x) = \overline{g(-x)}$.
    (a) Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(t)|^2\, dt = (g * \tilde{g})(0).$$

    (b) Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |(f*g)(x)|^2\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |(f*\tilde{g})(x)|^2\, dx.$$

31. Let $f \in L^1(S^1)$ satisfy $\hat{f}(|n|) = -\hat{f}(-|n|) \geq 0$ for all $n \in \mathbb{Z}$. Show that

$$\sum_{n>0} \frac{\hat{f}(n)}{n} < \infty.$$

32. If $\{K_n\}_{n=1}^{\infty}$ and $\{J_n\}_{n=1}^{\infty}$ are families of good kernels on S^1, show that $\{K_n * J_n\}_{n=1}^{\infty}$ is also a family of good kernels.
33. Suppose f is absolutely continuous on S^1 with $f' \in L^2(S^1)$. Prove that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| \leq \|f\|_1 + 2\sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}}\, \|f'\|_2.$$

34. Show that there exists a function $f \in L^1(S^1)$ for which the partial sums $S_n(f)$ of its Fourier series fail to converge to f in the L^1-norm.
35. Let $f \in L^1(S^1)$ and $S_n(f)$ denote the n-th partial sum of the Fourier series of f. Show that

$$\left\| \frac{S_n(f)}{n} \right\|_1 \to 0 \quad \text{as } n \to \infty.$$

36. If f is Riemann integrable on $[-\pi, \pi]$ and differentiable at $t_0 \in [-\pi, \pi]$, prove that

$$S_n(f; t_0) \to f(t_0) \quad \text{as } n \to \infty.$$

37. Suppose $f \in C^1(S^1)$ satisfies

$$(f * (1+f))(t) = f'(t)$$

    for all $t \in S^1$. Prove that f is a trigonometric polynomial.