MA15010H: Multi-variable Calculus

(Assignment 1 Hint/model solutions: Sequential continuity and vector differentiability)

July - November, 2025

1. Let $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Examine whether $A \cap B$ is (a) an open set (b) a closed set in \mathbb{R}^3 .

Solution. We have $(0,0,0) \in A \cap B$. If possible, let $(0,0,0) \in (A \cap B)^0$. Then there exists r > 0 such that $B_r((0,0,0)) \subseteq A \cap B$. Since $(0,0,\frac{r}{2}) \in B_r((0,0,0))$ but $(0,0,\frac{r}{2}) \notin A \cap B$, we get a contradiction. Hence $(0,0,0) \notin (A \cap B)^0$. Therefore $A \cap B$ is not an open set in \mathbb{R}^3 . Again, since $(1-\frac{1}{n},0,0) \in A \cap B$ for all $n \in \mathbb{N}$ and since $(1-\frac{1}{n},0,0) \to (1,0,0) \notin A \cap B$, $A \cap B$ is not a closed set in \mathbb{R}^3 .

2. Show that $\{x \in \mathbb{R}^m : 1 < ||x|| \le 2\}$ is neither an open set nor a closed set in \mathbb{R}^m .

Solution. Let $S = \{x \in \mathbb{R}^m : 1 < ||x|| \le 2\}$. Since $||(2 + \frac{1}{n})e_1|| = 2 + \frac{1}{n} > 2$ for all $n \in \mathbb{N}$, $(2 + \frac{1}{n})e_1 \in \mathbb{R}^m \setminus S$ for all $n \in \mathbb{N}$. Also, $(2 + \frac{1}{n})e_1 \to 2e_1 \notin \mathbb{R}^m \setminus S$, since $||2e_1|| = 2$. Hence $\mathbb{R}^m \setminus S$ is not a closed set in \mathbb{R}^m and consequently S is not an open set in \mathbb{R}^m .

Again, since $\|(1+\frac{1}{n})e_1\|=1+\frac{1}{n}\in(1,2]$ for all $n\in\mathbb{N}$, $(1+\frac{1}{n})e_1\in S$ for all $n\in\mathbb{N}$. Also, $(1+\frac{1}{n})e_1\to e_1\notin S$, since $\|e_1\|=1$. Hence S is not a closed set in \mathbb{R}^m .

3. State TRUE or FALSE with justification: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and if S is a bounded subset of \mathbb{R}^2 , then f(S) must be a bounded subset of \mathbb{R} .

Solution. Since S is a bounded subset of \mathbb{R}^2 , there exists r > 0 such that $S \subseteq B_r[0]$. Now, since $B_r[0]$ is a closed and bounded set in \mathbb{R}^2 and $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous, $f(B_r[0])$ is a bounded set in \mathbb{R} . Hence there exists M > 0 such that $|f(x)| \leq M$ for all $x \in B_r[0]$. So, in particular, $|f(x)| \leq M$ for all $x \in S$. Hence f(S) is a bounded subset of \mathbb{R} . Therefore the given statement is TRUE.

4. Let S be a nonempty subset of \mathbb{R}^m such that every continuous function $f: S \to \mathbb{R}$ is bounded. Show that S is a closed and bounded set in \mathbb{R}^m .

Solution. If possible, let S be not closed in \mathbb{R}^m . Then there exists $x_0 \in \mathbb{R}^m \setminus S$ and a sequence (x_n) in S such that $x_n \to x_0$. The function $f: S \to \mathbb{R}$, defined by $f(x) = \frac{1}{\|x - x_0\|}$ for all $x \in S$, is continuous but not bounded (since $\|x_n - x_0\| \to 0$ and so $f(x_n) \to \infty$), which contradicts the hypothesis. Hence S must be a closed set in \mathbb{R}^m .

Again, if possible, let S be not bounded in \mathbb{R}^m . Then the function $g: S \to \mathbb{R}$, defined by g(x) = ||x|| for all $x \in S$, is continuous but not bounded, which contradicts the hypothesis. Hence S must be bounded in \mathbb{R}^m .

5. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ and let $f : S \to \mathbb{R}$ be continuous. Show that there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$ such that $f(S) = [\alpha, \beta]$.

Solution. We know that $S = B_1[0]$ is a closed and bounded set in \mathbb{R}^3 . Since $f: S \to \mathbb{R}$ is continuous, there exist $x_0, y_0 \in S$ such that

$$f(x_0) \le f(x) \le f(y_0)$$
 for all $x \in S$.

Taking $\alpha = f(x_0)$ and $\beta = f(y_0)$, we find that $\alpha, \beta \in \mathbb{R}, \alpha \leq \beta$, and

$$f(S) \subseteq [\alpha, \beta].$$

Again, if $t \in [0, 1]$, then $(1 - t)x_0 + ty_0 \in \mathbb{R}^3$ and

$$||(1-t)x_0 + ty_0|| \le (1-t)||x_0|| + t||y_0|| \le 1 - t + t = 1,$$

so $(1-t)x_0 + ty_0 \in S$. Let

$$F(t) = (1 - t)x_0 + ty_0,$$

and

$$\varphi(t) = f(F(t)) = f((1-t)x_0 + ty_0)$$
 for all $t \in [0,1]$.

Since the functions $F:[0,1]\to S$ and $f:S\to\mathbb{R}$ are continuous, $\varphi=f\circ F:[0,1]\to\mathbb{R}$ is continuous.

Assuming $\alpha < \beta$, let $\gamma \in (\alpha, \beta) = (\varphi(0), \varphi(1))$. Then by the intermediate value property of the continuous function φ , there exists $t_0 \in (0,1)$ such that

$$\gamma = \varphi(t_0) = f(F(t_0)) \in f(S),$$

since $F(t_0) \in S$. Therefore $f(S) = [\alpha, \beta]$.

6 (a). Examine whether

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$$

exists (in \mathbb{R}) and find its value if it exists (in \mathbb{R}).

Solution. Let (x_n, y_n) be any sequence in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $(x_n, y_n) \to (0,0)$. Then $x_n^2 + y_n^2 \neq 0$ for all $n \in \mathbb{N}$ and $x_n^2 + y_n^2 \to 0$ in \mathbb{R} . Since

$$\lim_{t \to 0} \frac{1 - \cos t}{t^2} = \lim_{t \to 0} \frac{\sin t}{2t} = \frac{1}{2},$$

we have

$$\lim_{n \to \infty} \frac{1 - \cos(x_n^2 + y_n^2)}{(x_n^2 + y_n^2)^2} = \frac{1}{2}.$$

It follows that

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$$

exists and its value is $\frac{1}{2}$.

(b). Examine whether

$$\lim_{(x,y)\to(0,0)} \frac{y}{x^2+y^2} \sin\left(\frac{1}{x^2+y^2}\right)$$

exists (in \mathbb{R}) and find its value if it exists (in \mathbb{R}).

Solution. Let
$$f(x,y) = \frac{y}{x^2 + y^2} \sin\left(\frac{1}{x^2 + y^2}\right)$$
 for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}.$

Since

$$\left(0, \frac{\sqrt{2}}{\sqrt{(4n+1)\pi}}\right) \to (0,0) \quad \text{and} \quad f\left(0, \frac{\sqrt{2}}{\sqrt{(4n+1)\pi}}\right) = \sqrt{2n\pi + \frac{\pi}{2}} \to \infty,$$

the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (in \mathbb{R}).

7. Let S be a nonempty open set in \mathbb{R} and let $F: S \to \mathbb{R}^m$ be a differentiable function such that ||F(t)|| is constant for all $t \in S$. Show that

$$F(t) \cdot F'(t) = 0$$
 for all $t \in S$.

Solution. Let $c \in \mathbb{R}$ such that

$$||F(t)|| = c$$
 for all $t \in S$.

Then

$$F(t) \cdot F(t) = ||F(t)||^2 = c^2$$
 for all $t \in S$.

Hence

$$\frac{d}{dt}(F(t)\cdot F(t)) = 0$$
 for all $t\in S$.

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This gives

$$F'(t) \cdot F(t) + F(t) \cdot F'(t) = 0$$
 for all $t \in S$

So

$$2F(t) \cdot F'(t) = 0$$
 for all $t \in S$.

Therefore

$$F(t) \cdot F'(t) = 0$$
 for all $t \in S$.