MA15010H: Multi-variable Calculus

(Assignment 3 Hint/model solutions: Directional derivatives and differentiability)

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1. *Let S be a nonempty open subset of $\mathbb{R}^2$ and let $f : S \to \mathbb{R}$ be such that the partial derivatives f_x and f_y exist at each point of S. If $f_x : S \to \mathbb{R}$ and $f_y : S \to \mathbb{R}$ are bounded, then show that f is continuous.*

Solution. Since f_x and f_y are bounded, there exist $M_1, M_2 > 0$ such that $|f_x(x, y)| \le M_1$ and $|f_y(x, y)| \le M_2$ for all $(x, y) \in S$. Let $(x_0, y_0) \in S$. Since S is open in $\mathbb{R}^2$, there exists $r > 0$ such that $B_r((x_0, y_0)) \subseteq S$. For all $h, k \in \mathbb{R}$ with $|h| < \frac{r}{2}$, $|k| < \frac{r}{2}$, we have

$$\begin{aligned}
|f(x_0 + h, y_0 + k) - f(x_0, y_0)| &= |f(x_0 + h, y_0 + k) - f(x_0, y_0 + k) \\
&\quad + f(x_0, y_0 + k) - f(x_0, y_0)| \\
&\le |f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)| \\
&\quad + |f(x_0, y_0 + k) - f(x_0, y_0)| \\
&\le |h||f_x(x_0 + \theta_1 h, y_0 + k)| + |k||f_y(x_0, y_0 + \theta_2 k)|
\end{aligned}$$

for some $\theta_1, \theta_2 \in (0, 1)$ (using Lagrange's mean value theorem of single real variable). Hence if $\epsilon > 0$, then choosing $\delta = \min\left\{\frac{r}{2}, \frac{\epsilon}{M_1+M_2}\right\} > 0$, we find that $|f(x_0 + h, y_0 + k) - f(x_0, y_0)| \le M_1|h| + M_2|k| < \epsilon$ for all $(h, k) \in \mathbb{R}^2$ with $\|(h, k)\| = \sqrt{h^2 + k^2} < \delta$. Therefore f is continuous at (x_0, y_0). Since $(x_0, y_0) \in S$ is arbitrary, f is continuous. $\square$

2. *Find all $u \in \mathbb{R}^2$ with $\|u\| = 1$ for which the directional derivative $D_u f(0, 0)$ exists (in $\mathbb{R}$), if for all $(x, y) \in \mathbb{R}^2$,*

$$f(x, y) = \begin{cases} 1, & \text{if } y < x^2 < 2y, \\ 0, & \text{otherwise.} \end{cases}$$

Solution. Let $u = (u_1, u_2) \in \mathbb{R}^2$ with $\|u\| = 1$. We have

$$\lim_{t\to 0} \frac{f((0, 0) + tu) - f(0, 0)}{t} = \lim_{t\to 0} \frac{f(tu_1, tu_2)}{t} = \lim_{t\to 0} \frac{0}{t} = 0.$$

(The inequalities $tu_2 < t^2u_1^2 < 2tu_2$ are equivalent to the inequalities:

$$(i) u_2 < tu_1^2 < 2u_2 \text{ if } t > 0$$

$$(ii) u_2 > tu_1^2 > 2u_2 \text{ if } t < 0.$$

We can make $|tu_1^2|$ arbitrarily small for sufficiently small $|t| > 0$ and hence for such t, at least one inequality in each of (i) and (ii) cannot be satisfied. Thus we get $f(tu_1, tu_2) = 0$ for sufficiently small $|t| > 0$.) Therefore $D_u f(0,0)$ exists (and equals 0) for each $u \in \mathbb{R}^2$ with $\|u\| = 1$.

$\square$

3. *State TRUE or FALSE with justification: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous such that all the directional derivatives of f at $(0,0)$ exist (in $\mathbb{R}$), then f must be differentiable at $(0,0)$.*

Solution. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x^2y\sqrt{x^2+y^2}}{x^4+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

We know that f is continuous at each point of $\mathbb{R}^2 \setminus \{(0,0)\}$. Let $\epsilon > 0$. We have

$$|f(x,y) - f(0,0)| = \left|\frac{x^2y}{x^4+y^2}\right| \sqrt{x^2+y^2} \leq \frac{1}{2}\sqrt{x^2+y^2}$$

for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $|f(x,y) - f(0,0)| = 0$ if $(x,y) = (0,0)$. Hence choosing $\delta = 2\epsilon > 0$, we find that $|f(x,y) - f(0,0)| < \epsilon$ for all $(x,y) \in \mathbb{R}^2$ satisfying $\|(x,y) - (0,0)\| = \sqrt{x^2+y^2} < \delta$. This shows that f is continuous at $(0,0)$ and therefore f is continuous.

If $u = (u_1, u_2) \in \mathbb{R}^2$ with $\|u\| = 1$, then

$$\lim_{t\to 0} \frac{f((0,0)+tu) - f(0,0)}{t} = \lim_{t\to 0} \frac{u_1^2 u_2 t|t|\sqrt{u_1^2+u_2^2}}{t^2u_1^4 + u_2^2} = 0$$

i.e., $D_u f(0,0)$ exists. Hence all the directional derivatives of f at $(0,0)$ exist.

Again,

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{h^2k}{h^4+k^2} \neq 0.$$

since $\left(\frac{1}{n}, \frac{1}{n^2}\right) \to (0,0)$, but

$$\frac{\frac{1}{n^2}\frac{1}{n^2}}{\frac{1}{n^4}+\frac{1}{n^4}} = \frac{1}{2} \neq 0.$$

Hence f is not differentiable at $(0,0)$. Therefore the given statement is **FALSE**.

$\square$

4. *Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, if for all $(x, y) \in \mathbb{R}^2$,*

$$f(x,y) = \begin{cases} x^{4/3} \sin\left(\frac{y}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Solution. Let $E = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$. Since

$$f_x(x,y) = \frac{4}{3}x^{1/3} \sin\left(\frac{y}{x}\right) - \frac{y}{x^{2/3}} \cos\left(\frac{y}{x}\right) \quad \text{and} \quad f_y(x,y) = x^{1/3} \cos\left(\frac{y}{x}\right)$$

for all $(x, y) \in E$. $f_x : E \to \mathbb{R}$ and $f_y : E \to \mathbb{R}$ are continuous. Hence f is differentiable at all $(x, y) \in E$. Let $y_0 \in \mathbb{R}$ and let $\epsilon > 0$. Then

$$f_x(0, y_0) = \lim_{h \to 0} \frac{f(h, y_0) - f(0, y_0)}{h} = \lim_{h \to 0} h^{1/3} \sin\left(\frac{y_0}{h}\right) = 0$$

(since $|h^{1/3} \sin(\frac{y_0}{h})| \leq |h|^{1/3}$ for all $h \in \mathbb{R} \backslash \{0\}$) and

$$f_y(0, y_0) = \lim_{k \to 0} \frac{f(0, y_0 + k) - f(0, y_0)}{k} = 0.$$

Also, for all $(x, y) \in E$, we have $f_y(x, y) = x^{1/3} \cos(y/x)$, and so

$$|f_y(x, y) - f_y(0, y_0)| \leq |x|^{1/3} < \epsilon \quad \text{for all } (x, y) \in B_\delta((0, y_0)),$$

where $\delta = \epsilon^3 > 0$. Thus $f_x(0, y_0)$ exists (in $\mathbb{R}$), $f_y(x, y)$ exists (in $\mathbb{R}$) for all $(x, y) \in \mathbb{R}^2$ and $f_y : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0, y_0)$. Hence by Ex.21 of Practice Problem Set - 3, f is differentiable at $(0, y_0)$. Therefore f is differentiable at all points of $\mathbb{R}^2$. $\square$

Alternative solution: As shown above, f is differentiable at all $(x, y) \in \mathbb{R}^2$ for which $x \neq 0$. Let $y_0 \in \mathbb{R}$. Then as shown above, $f_x(0, y_0) = f_y(0, y_0) = 0$. For all $(h, k) \in \mathbb{R}^2$ with $h \neq 0$, we have

$$\begin{aligned} \epsilon(h,k) = \frac{|f(h, y_0 + k) - f(0, y_0) - hf_x(0, y_0) - kf_y(0, y_0)|}{\sqrt{h^2 + k^2}} &= \frac{h^{4/3} \sin|\left(\frac{y_0+k}{h}\right)|}{\sqrt{h^2 + k^2}} \\ &= \frac{|h|^{1/3}|h||\sin(\frac{y_0+k}{h})|}{\sqrt{h^2 + k^2}} \leq |h|^{1/3}. \end{aligned}$$

Also, $\epsilon(0, k) = 0$ for all $k \in \mathbb{R} \setminus \{0\}$. Hence it follows that

$$\lim_{(h,k) \to (0,0)} \epsilon(h, k) = 0.$$

Consequently, f is differentiable at $(0, y_0)$. Therefore f is differentiable at all points of $\mathbb{R}^2$.

5. *Let $f : S \subseteq \mathbb{R}^m \to \mathbb{R}$ be differentiable at $x_0 \in S^0$ and let $f(x_0) = 0$. If $g : S \to \mathbb{R}$ is continuous at x_0, then show that $fg : S \to \mathbb{R}$, defined by $(fg)(x) = f(x)g(x)$ for all $x \in S$, is differentiable at x_0.*

Solution. Since f is differentiable at x_0, there exists $\alpha \in \mathbb{R}^m$ such that
$$\lim_{h\to 0} \frac{|f(x_0+h) - f(x_0) - \alpha \cdot h|}{\|h\|} = 0.$$
For all $h \in \mathbb{R}^m$ for which $x_0 + h \in S$, we have
$$(fg)(x_0+h)-(fg)(x_0)-g(x_0)\alpha\cdot h = (f(x_0+h)-f(x_0)-\alpha\cdot h)g(x_0+h)+(g(x_0+h)-g(x_0))\alpha\cdot h.$$
Hence for all $h \in \mathbb{R}^m \setminus \{0\}$ for which $x_0 + h \in S$, we have
$$\begin{aligned}\frac{|(fg)(x_0+h) - (fg)(x_0) - g(x_0)\alpha \cdot h|}{\|h\|} &\le \frac{|f(x_0+h) - f(x_0) - \alpha \cdot h|}{\|h\|}|g(x_0+h)| \\ &\quad + |g(x_0+h) - g(x_0)|\frac{|\alpha\cdot h|}{\|h\|}.\end{aligned}$$
Since g is continuous at x_0, $\lim_{h\to 0} g(x_0+h) = g(x_0)$ and since $|\alpha \cdot h| \le \|\alpha\|\|h\|$, it follows that
$$\lim_{h\to 0} \frac{|(fg)(x_0+h) - (fg)(x_0) - g(x_0)\alpha \cdot h|}{\|h\|} = 0.$$
Since $g(x_0)\alpha \in \mathbb{R}^m$, we conclude that fg is differentiable at x_0. $\square$

6. *Show that $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(x_0, y_0) \in S^0$ if and only if there exist functions $\varphi, \psi : S \to \mathbb{R}$ such that φ, ψ are continuous at (x_0, y_0) and*
$$f(x, y) - f(x_0, y_0) = (x - x_0)\varphi(x, y) + (y - y_0)\psi(x, y)$$
for all $(x, y) \in S$.

Solution. We first assume that f is differentiable at (x_0, y_0). Then $\alpha = f_x(x_0, y_0)$ and $\beta = f_y(x_0, y_0)$ exist (in $\mathbb{R}$). For each $(x, y) \in S$, let
$$g(x, y) = f(x, y) - f(x_0, y_0) - \alpha(x - x_0) - \beta(y - y_0),$$
then define
$$\varphi(x,y) = \begin{cases} \alpha + \dfrac{(x-x_0)g(x,y)}{(x-x_0)^2+(y-y_0)^2}, & (x,y) \ne (x_0,y_0), \\ \alpha, & (x,y) = (x_0,y_0), \end{cases}$$
and
$$\psi(x,y) = \begin{cases} \beta + \dfrac{(y-y_0)g(x,y)}{(x-x_0)^2+(y-y_0)^2}, & (x,y) \ne (x_0,y_0), \\ \beta, & (x,y) = (x_0,y_0). \end{cases}$$
If $(x, y) \in S \setminus \{(x_0, y_0)\}$, then
$$(x-x_0)\varphi(x,y)+(y-y_0)\psi(x,y) = \alpha(x-x_0)+\beta(y-y_0)+g(x,y) = f(x,y)-f(x_0,y_0).$$

Also, if $(x,y) = (x_0,y_0)$, then

$$(x-x_0)\varphi(x,y) + (y-y_0)\psi(x,y) = 0 = f(x,y) - f(x_0,y_0).$$

Hence for all $(x,y) \in S$, $f(x,y) - f(x_0,y_0) = (x-x_0)\varphi(x,y) + (y-y_0)\psi(x,y)$.

Again, for all $(x,y) \in S \setminus \{(x_0,y_0)\}$, we have

$$|\varphi(x,y) - \varphi(x_0,y_0)| = \frac{|x-x_0||g(x,y)|}{(x-x_0)^2 + (y-y_0)^2} \le \frac{|g(x,y)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}.$$

Since f is differentiable at (x_0,y_0), the limit

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|g(x,y)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0,$$

and hence it follows that

$$\lim_{(x,y)\to(x_0,y_0)} \varphi(x,y) = \varphi(x_0,y_0).$$

Therefore φ is continuous at (x_0,y_0). Similarly, we can show ψ is continuous at (x_0,y_0).

Conversely, let there exist functions $\varphi, \psi : S \to \mathbb{R}$ such that φ, ψ are continuous at (x_0,y_0) and

$$f(x,y) - f(x_0,y_0) = (x-x_0)\varphi(x,y) + (y-y_0)\psi(x,y)$$

for all $(x,y) \in S$. Then for all $(x,y) \in S \setminus \{(x_0,y_0)\}$, we have

$$\begin{aligned}
&\frac{|f(x,y) - f(x_0,y_0) - (x-x_0)\varphi(x_0,y_0) - (y-y_0)\psi(x_0,y_0)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \\
&\le \frac{(x-x_0)|\varphi(x,y) - \varphi(x_0,y_0)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} + \frac{(y-y_0)|\psi(x,y) - \psi(x_0,y_0)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \\
&\le |\varphi(x,y) - \varphi(x_0,y_0)| + |\psi(x,y) - \psi(x_0,y_0)|.
\end{aligned}$$

Since φ and ψ are continuous at (x_0,y_0),

$$\lim_{(x,y)\to(x_0,y_0)} |\varphi(x,y) - \varphi(x_0,y_0)| = 0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} |\psi(x,y) - \psi(x_0,y_0)| = 0.$$

Hence,

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|f(x,y) - f(x_0,y_0) - (x-x_0)\varphi(x_0,y_0) - (y-y_0)\psi(x_0,y_0)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0,$$

and therefore f is differentiable at (x_0,y_0). $\square$

7. *Let the temperature $T(x,y)$ at any point $(x,y) \in \mathbb{R}^2$ be given by $T(x,y) = 2x^2 + xy + y^2$. An insect is at the point $(1,1)$.*
(a) What is the best direction for the insect to move to feel cooler?
(b) In which direction should the insect move to feel no change in temperature?

Solution. Since $T_x(x,y) = 4x + y$ and $T_y(x,y) = x + 2y$ for all $(x,y) \in \mathbb{R}^2$, $T_x : \mathbb{R}^2 \to \mathbb{R}$ and $T_y : \mathbb{R}^2 \to \mathbb{R}$ are continuous and hence $T : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. Since

$$\nabla T(1,1) = (T_x(1,1), T_y(1,1)) = (5,3),$$

the temperature will decrease fastest in the direction

$$-\frac{1}{\|\nabla T(1,1)\|}\nabla T(1,1) = \left(-\frac{5}{\sqrt{34}}, -\frac{3}{\sqrt{34}}\right),$$

and so this is the best direction for the insect to start moving to feel cooler.

Again, if $u = (u_1, u_2) \in \mathbb{R}^2$ with $\|u\| = 1$ is the direction for the insect to feel no change in temperature, then we must have

$$D_u T(1,1) = \nabla T(1,1) \cdot u = 0.$$

This gives $5u_1 + 3u_2 = 0$. Since we also have $u_1^2 + u_2^2 = 1$, we get

$$u = \left(\frac{3}{\sqrt{34}}, -\frac{5}{\sqrt{34}}\right) \quad \text{or} \quad \left(-\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\right).$$

$\square$