MA201: Complex Analysis

Assignment 1

(Elementary properties of complex numbers)

July - November, 2024

1. Prove the following statements:
   (a) If $z \in \mathbb{C}$, then $|z| \leq |\Re(z)| + |\mathrm{Im}(z)| \leq \sqrt{2}\,|z|$.
   (b) If $z_1, z_2 \in \mathbb{C}$, then $|z_1 + z_2| \leq |z_1| + |z_2|$. Show that equality holds if and only if one of them is a nonnegative scalar multiple of the other.
   (c) If either $|z_1| = 1$ or $|z_2| = 1$, but not both, then prove that $\left|\dfrac{z_1 - z_2}{1 - \overline{z_1}\, z_2}\right| = 1$. What exception must be made for the validity of the above equality when $|z_1| = |z_2| = 1$?
2. Show that the equation $z^4 + z + 5 = 0$ has no solution in the set $\{z \in \mathbb{C} \colon \; |\, z\, |< 1\}$.
3. If z and w are in $\mathbb{C}$ such that $\mathrm{Im}(z) > 0$ and $\mathrm{Im}(w) > 0$, show that $|\, \frac{z-w}{z-\overline{w}}\, |< 1$.
4. When does $az + b\overline{z} + c = 0$ has exactly one solution?
5. If $1 = z_0, z_1, \ldots, z_{n-1}$ are distinct n^{th} roots of unity, prove that

$$\prod_{j=1}^{n-1}(z - z_j) = \sum_{j=0}^{n-1} z^j.$$