MA201: Complex Analysis

(Assignment 1 Hint/ model solutions: Elementary properties of complex numbers)

July - November, 2024

1. Prove the following statements:

   (a) If $z \in \mathbb{C}$, then $|z| \leq |\Re(z)| + |\mathrm{Im}(z)| \leq \sqrt{2}\,|z|$.

   Answer: Observe that, for any two real numbers x and y, we have

$$(|x| - |y|)^2 \geq 0 \Longrightarrow |x|^2 + |y|^2 \geq 2|x|\,|y|.$$

   Let $z = x + iy$ be any complex number.

   Now,

$$\begin{aligned}(|x| + |y|)^2 &= |x|^2 + |y|^2 + 2|x|\,|y| \leq |x|^2 + |y|^2 + |x|^2 + |y|^2 \\ &= 2\left(|x|^2 + |y|^2\right) = 2\left(x^2 + y^2\right) = 2|z|^2 \ .\end{aligned}$$

   This gives that $|\Re(z)| + |\mathrm{Im}(z)| = |x| + |y| \leq \sqrt{2}\,|z|$.

$$\begin{aligned}|z| = \sqrt{x^2 + y^2} \leq \sqrt{x^2 + y^2 + 2|x|\,|y|} &= \sqrt{|x|^2 + |y|^2 + 2|x|\,|y|} \\ &= \sqrt{(|x| + |y|)^2} = |x| + |y| \ .\end{aligned}$$

   That is, $|z| \leq |\Re(z)| + |\mathrm{Im}(z)|$.

   (b) If $z_1, z_2 \in \mathbb{C}$, then $|z_1 + z_2| \leq |z_1| + |z_2|$. Show that equality holds if and only if one of them is a nonnegative scalar multiple of the other.

   Answer: Do yourself.

   (c) If either $|z_1| = 1$ or $|z_2| = 1$, but not both, then prove that $\left|\dfrac{z_1 - z_2}{1 - \overline{z_1}\, z_2}\right| = 1$. What exception must be made for the validity of the above equality when $|z_1| = |z_2| = 1$?

   Answer: Case I: $|z_1| = 1$ and $|z_2| \neq 1$

$$\begin{aligned}\left|\frac{z_1 - z_2}{1 - \overline{z_1}\, z_2}\right|^2 &= \frac{|z_1|^2 + |z_2|^2 - 2\Re(z_1\overline{z_2})}{1 + |\overline{z_1}z_2|^2 - 2\Re(z_1\overline{z_2})} \\ &= \frac{1 + |z_2|^2 - 2\Re(z_1\overline{z_2})}{1 + |z_2|^2 - 2\Re(z_1\overline{z_2})} = 1 \ .\end{aligned}$$

   Observe that the denominator $1 + |\overline{z_1}z_2|^2 - 2\Re(z_1\overline{z_2}) \neq 0$ if $|z_1| = 1$ and $|z_2| \neq 1$.

   Case II: $|z_2| = 1$ and $|z_1| \neq 1$

   It can be worked out similarly as in the previous case.

   Case III: $|z_1| = 1$ and $|z_2| = 1$

Then, $\left|\dfrac{z_1 - z_2}{1-\overline{z_1}\, z_2}\right|^2 = \dfrac{2-2\Re(z_1\overline{z_2})}{2-2\Re(z_1\overline{z_2})} = 1$ if the denominator $2-2\Re(z_1\overline{z_2}) \neq 0$. That is, $\Re(z_1\overline{z_2}) \neq 1$ if and only if $z_1 \neq z_2$. So, the exception is to be made for the validity of the above equality in this case is $z_1 \neq z_2$.

2. Show that the equation $z^4 + z + 5 = 0$ has no solution in the set $\{z \in \mathbb{C}\colon\ |z| < 1\}$.

   Answer: Suppose α is a solution. So $|\alpha| < 1$ and $\alpha^4 + \alpha = -5$. Then $5 = |\alpha^4 + \alpha| \leq 2$.

3. If z and w are in $\mathbb{C}$ such that $\mathrm{Im}(z) > 0$ and $\mathrm{Im}(w) > 0$, show that $|\frac{z-w}{z-\overline{w}}| < 1$.

   Answer:
$$\left|\frac{z-w}{z-\bar{w}}\right|^2 \leq 1 \Longleftrightarrow (z-\bar{z})(w-\bar{w}) < 0.$$
   Clearly $(z-\bar{z})(w-\bar{w}) < 0$ if $\mathrm{Im}(z) > 0$ and $\mathrm{Im}(w) > 0$.

4. When does $az + b\overline{z} + c = 0$ has exactly one solution?

   Answer: Let $z = x+iy, a = a_1 + ia_2, b = b_1 + ib_2$ and $c = c_1 + ic_2$ and put these values in the given equation $az + b\overline{z} + c = 0$. After simplification (please check it carefully!) we have,
$$(a_1x - a_2y) + i(a_2x + a_1y) + (b_1x + b_2y) + i(b_2x - b_1y) + c_1 + ic_2 = 0$$
   After equating real and imaginary parts we get the following system of linear equations
$$(a_1 + b_1)x + (b_2 - a_2)y = c_1$$
$$(a_2 + b_2)x + (a_1 - b_1)y = c_2.$$
   Therefore given equation has exactly one solution if the above system of linear equations has unique solution. In this case
$$(a_1 + b_1)(a_1 - b_1) - (b_2 - a_2)(a_2 + b_2) \neq 0.$$
   In fact the given equation has exactly one solution if $|a| \neq |b|$.

5. If $1 = z_0, z_1, \ldots, z_{n-1}$ are distinct n^{th} roots of unity, prove that
$$\prod_{j=1}^{n-1}(z - z_j) = \sum_{j=0}^{n-1} z^j.$$
   Answer: The points $1 = z_0, z_1, \ldots, z_{n-1}$ are the roots of $z^n - 1 = 0$. So $z^n - 1 = (z-1)\Pi_{j=1}^{n-1}(z - z_j) = (z-1)\sum_{j=0}^{n-1} z^j$. Let $f(z) = \Pi_{j=1}^{n-1}(z - z_j)$ and $g(z) = \sum_{j=0}^{n-1} z^j$. Thus $f(z) = g(z)$ for $z \neq 1$. Since both f, g are continuous it follows that $f(1) = g(1)$ as well.