MA201: Complex Analysis

Assignment 2

(Topology (open set, closed set, etc) of complex plane and differentiability)

July - November, 2024

1. For each subset of $\mathbb{C}$, determine if it is open, closed, or not, with justification:
   (a) $A_1 = \{z \in \mathbb{C} : Re(z) = 1 \text{ and } Im(z) \neq 4\}$
   (b) $A_2 = B(1,1) \cup B(2, \frac{1}{2}) \cup B(3, \frac{1}{3})$
   (c) $A_3 = \left\{z \in \mathbb{C} : \left|\frac{z-1}{z+1}\right| = 2\right\}$
   (d) $A_4 = \{z \in \mathbb{C} : \sin(Re(z)) < Im(z) < 1\}$.
2. For each of the following subsets of $\mathbb{C}$, determine their interior, exterior and boundary:
   (a) $S_1 = \{z \in \mathbb{C} : |z| < 1 \text{ and } Im(z) \neq 0\} \cup \{z \in \mathbb{C} : |z| > 1 \text{ and } Im(z) = 0\}$
   (b) $S_2 = \left\{r\left(\cos\left(\frac{1}{n}\right) + i\sin\left(\frac{1}{n}\right)\right) \in \mathbb{C} : r > 0, n \in \mathbb{N}\right\} \cup \{z \in \mathbb{C} : Re(z) < 0\}$.
3. Show that $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,\ y) = (x,\ -y)$ for all $(x,\ y) \in \mathbb{R}^2$ is differentiable at every point in $\mathbb{R}^2$. View the same function as a complex function. Show that $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \overline{z}$ for all $z \in \mathbb{C}$ is not differentiable at any point in $\mathbb{C}$.
4. Let $f(z) = z^3$. For $z_1 = 1$ and $z_2 = i$, show that there does not exist any point c on the line $y = 1 - x$ joining z_1 and z_2 such that
   $$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = f'(c)$$
   (The mean value theorem does not extend to complex derivatives).
5. If $f(z)$ is a real valued function in a domain $D \subseteq \mathbb{C}$, then show that either $f'(z) = 0$ or $f'(z)$ does not exist in D.
6. Let U be an open set and $f\colon U \to \mathbb{C}$ be a differentiable function. Let $\overline{U} := \{\overline{z}\colon z \in U\}$. Show that the function $g\colon \overline{U} \to \mathbb{C}$ defined by $g(z) := \overline{f(\overline{z})}$ is differentiable on $\overline{U}$.
7. Derive the Cauchy-Riemann equations in polar coordinates.
8. Let $f : \mathbb{D} \to \mathbb{C}$ be a differentiable function such that, for all $z, w \in \mathbb{C}, f(z) = f(w)$, whenever $|z| = |w|$. Prove that f is a constant function. (Use CR equations in polar coordinates)
9. Let $f = u + iv$ is an analytic function on the whole complex plane $\mathbb{C}$. If $u(x,y) = \phi(x)$ and $v(x,y) = \psi(y)$ prove that, for all $z \in \mathbb{C}$, $f(z) = az + b$ for some $a, b \in \mathbb{C}$.