MA201: Complex Analysis

(Assignment 2 Hint/ model solutions: Topology of complex plane and differentiability)

July - November, 2024

1. For each subset of $\mathbb{C}$, determine if it is open, closed, or not, with justification:
   (a) $A_1 = \{z \in \mathbb{C} : Re(z) = 1 \text{ and } Im(z) \neq 4\}$
   Answer: $A_1 = \{z = x + iy : x = 1\} \setminus \{1 + 4i\}$. Neither open nor closed.
   (b) $A_2 = B(1, 1) \cup B(2, \frac{1}{2}) \cup B(3, \frac{1}{3})$
   Answer: Open set.
   (c) $A_3 = \left\{z \in \mathbb{C} : \left|\frac{z-1}{z+1}\right| = 2\right\}$
   Answer: $A_3 = \{z = x + iy : (x + 5/3)^2 + y^2 = 16/9\}$ is a closed set.
   (d) $A_4 = \{z \in \mathbb{C} : \sin(Re(z)) < Im(z) < 1\}$
   Answer: Let $S_1 = \{z \in \mathbb{C} : \sin(Re(z)) < Im(z)\}$ and $S_2 = \{z \in \mathbb{C} : Im(z) < 1\}$. Clearly both S_1 and S_2 are open sets. Hence the given set is open, being the intersection of S_1 and S_2.

2. For each of the following subsets of $\mathbb{C}$, determine their interior, exterior and boundary:
   (a) $S_1 = \{z \in \mathbb{C} : |z| < 1 \text{ and } Im(z) \neq 0\} \cup \{z \in \mathbb{C} : |z| > 1 \text{ and } Im(z) = 0\}$.
   Answer: Interior of $S_1 = \{z \in \mathbb{C} : |z| < 1 \text{ and } Im(z) \neq 0\}$.
   Exterior of $S_1 = (\bar{S}_1)^c = \emptyset$.
   Boundary of S_1 is the unit circle union the real line.
   (b) $S_2 = \left\{re^{i/n} \in \mathbb{C} : r > 0, n \in \mathbb{N}\right\} \cup \{z \in \mathbb{C} : Re(z) < 0\}$.
   Answer: Interior of $S_2 = \{z \in \mathbb{C} : Re(z) < 0\}$.
   Exterior of $S_2 = \bar{S}_2^c = \left[\left\{re^{i/n} \in \mathbb{C} : r > 0, n \in \mathbb{N}\right\} \cup \{z \in \mathbb{C} : Re(z) \leq 0\} \cup \{z \in \mathbb{C} : Re(z) \geq 0,\ Im(z) = 0\}\right]^c$.
   Boundary of $S_2 = \left\{re^{i/n} \in \mathbb{C} : r > 0, n \in \mathbb{N}\right\} \cup \{$ positive real axis $\} \cup \{$ imaginary axis $\}$.

3. Show that $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,\ y) = (x,\ -y)$ for all $(x,\ y) \in \mathbb{R}^2$ is differentiable at every point in $\mathbb{R}^2$. View the same function as a complex function. Show that $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \overline{z}$ for all $z \in \mathbb{C}$ is not differentiable at any point in $\mathbb{C}$.
   Answer: Do it yourself.

4. Let $f(z) = z^3$. For $z_1 = 1$ and $z_2 = i$, show that there does not exist any point c on the line $y = 1 - x$ joining z_1 and z_2 such that

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = f'(c)$$

   (Mean value theorem does not extend to complex derivatives).
   Answer: $\left|\dfrac{f(1) - f(i)}{1 - i}\right| = |\frac{1+i}{1-i}| = 1$. Any point on $[1, i]$ (is the line segment joining 1 and i) has mod value $\geq \frac{1}{\sqrt{2}}$. So $|f'(z)| = |3z^2| \geq \frac{3}{2} > 1$.

5. If $f(z)$ is a real valued function in a domain $D \subseteq \mathbb{C}$, then show that either $f'(z) = 0$ or $f'(z)$ does not exist in D.
   Hint: Use C-R equations.

Answer:

$$\lim_{w\to w_0\in\bar{U}} \frac{g(w)-g(w_0)}{w-w_0} = \lim_{z\to z_0\in U} \frac{g(\bar{z})-g(\bar{z}_0)}{\bar{z}-\bar{z}_0} = \lim_{z\to z_0\in U} \overline{\frac{f(z)-f(z_0)}{z-z_0}} = \overline{f'(z_0)}$$

7. Derive the Cauchy-Riemann equations in polar coordinates.

Answer: Let $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ be differentiable at $z_0 = r_0 e^{i\theta_0}$. Then

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

exists.

First we calculate the limit $z \to z_0$ for fixed θ and letting $r \to r_0$. Then

$$\begin{aligned} f'(z_o) &= \lim_{r\to r_0} \frac{f(re^{i\theta_0}) - f(r_0e^{i\theta_0})}{re^{i\theta_0} - r_0e^{i\theta_0}} \\ &= \frac{1}{e^{i\theta_0}} \lim_{r\to r_0} \frac{u(r,\theta_0) - u(r_0,\theta_0) + i[v(r,\theta_0) - v(r_0,\theta_0)]}{r-r_0} \\ &= \frac{1}{e^{i\theta_0}} \left(\frac{\partial u}{\partial r}(r_0,\theta_0) + i\frac{\partial v}{\partial r}(r_0,\theta_0) \right) \end{aligned}$$

Now calculate the limit $z \to z_0$ along the circle $r \to r_0$. In this case we have:

$$\begin{aligned} f'(z_o) &= \lim_{\theta\to\theta_0} \frac{f(r_0e^{i\theta}) - f(r_0e^{i\theta_0})}{r_0e^{i\theta} - r_0e^{i\theta_0}} \\ &= \frac{1}{r_0} \lim_{\theta\to\theta_0} \frac{u(r_0,\theta) - u(r_0,\theta_0) + i[v(r_0,\theta) - v(r_0,\theta_0)]}{e^{i\theta} - e^{i\theta_0}} \\ &= \frac{1}{r_0} \lim_{\theta\to\theta_0} \left\{ \left(\frac{u(r_0,\theta) - u(r_0,\theta_0) + i[v(r_0,\theta) - v(r_0,\theta_0)]}{\theta-\theta_0} \right) \frac{\theta-\theta_0}{e^{i\theta} - e^{i\theta_0}} \right\} \\ &= \frac{1}{ir_0e^{i\theta_0}} \left(\frac{\partial u}{\partial \theta}(r_0,\theta_0) + i\frac{\partial v}{\partial \theta}(r_0,\theta_0) \right) = \frac{1}{r_0e^{i\theta_0}} \left(\frac{\partial v}{\partial \theta}(r_0,\theta_0) - i\frac{\partial u}{\partial \theta}(r_0,\theta_0) \right) \end{aligned}$$

8. Let $f : \mathbb{D} \to \mathbb{C}$ be a differentiable function such that, for all $z, w \in \mathbb{C}, f(z) = f(w)$, whenever $|z| = |w|$. Prove that f is a constant function.

Hint: It is given that $f(z) = f(w)$ if $|z| = |w|$. This means that the function f is independent of the argument (i.e. $f(e^{i\theta}z) = f(z)$ for all θ.) Now, use C–R equations in the polar coordinates.

9. Let $f = u + iv$ is an analytic function defined on the whole complex plane $\mathbb{C}$. If $u(x,y) = \phi(x)$ and $v(x,y) = \psi(y)$ prove that, for all $z \in \mathbb{C}$, $f(z) = az + b$ for some $a \in \mathbb{C}$, $b \in \mathbb{C}$.

Answer: From C–R equations we have $\phi'(x) = \psi'(y)$ for all $z = x + iy \in \mathbb{C}$. In particular $\phi'(0) = \psi'(y)$ and $\phi'(x) = \psi'(0)$ for all $x, y \in \mathbb{R}$. Also $f'(z) = \phi'(x) = \psi'(y)$ hence $f'(z) = a(constant)$. If we consider $g(z) = f(z) - az$, then $g'(z) = 0$. Therefore, $g(z) = b(constant)$ i.e. $f(z) = az + b$.