MA201: Complex Analysis

Assignment 3

(Elementary Functions and Complex Line Integrals.)

July - November, 2024

1. Let u and v be nonconstant harmonic functions on $\mathbb{C}$.
   (a) If $U(x,y) = u(x,-y)$, is U also harmonic?
   (b) If v is a harmonic conjugate of u, is u a harmonic conjugate of v?
   (c) Is uv always harmonic? If not, produce an example.
2. If v is a harmonic conjugate of u (u, v real valued), prove that the functions uv and $u^2 - v^2$ are also harmonic.
3. What are all real valued harmonic functions u on D such that u^2 is also harmonic?
4. Find a harmonic conjugate, if it exists, of the following functions:
   (a) $u(x,y) = 2xy$.
   (b) $u(r,\theta) = r^n \cos n\theta$, $n \in \mathbb{N}$.
   (c) $u(x,y) = x^2 - y^2 + x + y - \frac{y}{x^2+y^2}$.
5. Let us define differential operators $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. Let $f = u + iv$ be defined on an open set in $\mathbb{C}$. Show that:
   (a) f satisfies C-R equations if and only if $\frac{\partial}{\partial \bar{z}} f(z) = 0$.
   (b) If $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, then show that $\Delta f = 4\frac{\partial^2}{\partial z \partial \bar{z}} f$.
   (c) Prove that the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}^n$ is harmonic for all $n \in \mathbb{N}$.
6. Find the values of z such that (a) $e^z \in \mathbb{R}$ and (b) $e^z \in i\mathbb{R}$.
7. Prove that $\sinh(\text{Im}z) \leq |\sin(z)| \leq \cosh(\text{Im}z)$. Deduce that $|\sin(z)|$ tends to ∞ as $|\text{Im}z| \to \infty$.
8. Find all the complex numbers which satisfy the following:
   (i) $\exp(z) = 1$ (ii) $\exp(z) = i$ (iii) $\exp(z-1) = 1$.
9. Evaluate the following:
   (i) $\log(3 - 2i)$ (ii) $\text{Log}\, i$ (iii) $(i)^{(-i)}$
10. If γ is the boundary of the triangle with vertices at the points 0, $3i$ and -4 oriented in the counterclockwise direction then show that $\left| \int_\gamma (e^z - \overline{z})\ dz \right| \leq 60$.
11. Evaluate $\int_\gamma |z|\ \overline{z}\ dz$ where γ is the circle $|z| = 2$.