MA201: Complex Analysis

Assignment 3

(Elementary Functions and Complex Line Integrals.)

July - November, 2024

1. Let u and v be nonconstant harmonic functions on $\mathbb{C}$.
   (a) If $U(x,y) = u(x,-y)$, is U also harmonic?

   Answer: Do it yourself.

   (b) If v is a harmonic conjugate of u, is u a harmonic conjugate of v?

   Answer: If v is a harmonic conjugate of u, and u a harmonic conjugate of v, then $f_1 = u + iv$ and $f_2 = v + iu$ both are analytic, which forces u and v to be constant.

   (c) Is uv always harmonic? If not, produce an example.

   Answer: Notice that uv is harmonic if and only if $u_x v_x + u_y v_y = 0$. In particular, $u(x,y) = x = v(x,y)$. Then uv is not harmonic.

2. If v is a harmonic conjugate of u (u, v real valued), prove that the functions uv and $u^2 - v^2$ are also harmonic.

   Answer: Let $f = u+iv$. Then f is analytic, and hence $f^2 = (u^2-v^2)+i(2uv)$ analytic. Implies uv and $u^2 - v^2$ are also harmonic.

3. What are all real valued harmonic functions u on D such that u^2 is also harmonic?

   Answer: u^2 is harmonic if and only if $u_x^2 + u_y^2 = 0$.

4. Find a harmonic conjugate, if it exists, of the following functions:
   (a) $u(x,y) = 2xy$.

   (b) $u(r,\theta) = r^n \cos n\theta$, $n \in \mathbb{N}$.

   (c) $u(x,y) = x^2 - y^2 + x + y - \frac{y}{x^2+y^2}$.

   Answer: (a) $v(x,y) = y^2 - x^2 + c$ (b) $v(r,\theta) = r^n \sin n\theta + c$

   (c) $v(x,y) = 2xy + y - x + \frac{x}{x^2+y^2} + c$.

5. Let us define differential operators $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. Let $f = u + iv$ be defined on an open set in $\mathbb{C}$. Show that:
   (a) f satisfies C-R equations if and only if $\frac{\partial}{\partial \bar{z}} f(z) = 0$.

   (b) If $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, then show that $\Delta f = 4\frac{\partial^2}{\partial z \partial \bar{z}} f$.

   (c) Prove that the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}^n$ is harmonic for all $n \in \mathbb{N}$.

   Answer: Do yourself.

6. Find the values of z such that (a) $e^z \in \mathbb{R}$ and (b) $e^z \in i\mathbb{R}$.

   Answer:

$$e^z = e^{x+iy} = e^x \ (\cos(y) \ + \ i \ \sin(y)) \ .$$

We know that $e^x \neq 0$ for all $x \in \mathbb{R}$.

e^z is a real number iff $\sin y = 0$ iff $y = n\pi$ where $n \in \mathbb{Z}$.

e^z is a pure imaginary number iff $\cos y = 0$ iff $y = \frac{(2n+1)\pi}{2}$ where $n \in \mathbb{Z}$.

7. Prove that $\sinh(\text{Im}z) \leq |\sin(z)| \leq \cosh(\text{Im}z)$. Deduce that $|\sin(z)|$ tends to ∞ as $|\text{Im}z| \to \infty$.

Answer: Set $z = x + i\,y$. We know that

$$\sin z = \sin x \ \cosh y \ + \ i \ \cos x \ \sinh y \ .$$

$$\begin{aligned} |\sin z|^2 &= \sin^2 x \ \cosh^2 y \ + \ \cos^2 x \ \sinh^2 y \\ &= \sin^2 x \ (1 + \sinh^2 y) \ + \ \cos^2 x \ \sinh^2 y \\ &= \sin^2 x \ + \ \sinh^2 y \end{aligned}$$

$$|\sin z| = \sqrt{\sin^2 x \ + \ \sinh^2 y} \leq \sqrt{1 \ + \ \sinh^2 y} = \sqrt{\cosh^2 y} = \cosh y \ .$$

Thus,

$$|\sin z| \leq \cosh y \ .$$

Now,

$$|\sin z|^2 = \sin^2 x \ + \ \sinh^2 y \geq \sinh^2 y = |\sinh y|^2 \ .$$

$$|\sin z| \geq |\sinh y| \geq \sinh y \ .$$

We know that $\sinh y$ tends to $+\infty$ as $y \to +\infty$ and $\sinh y$ tends to $-\infty$ as $y \to -\infty$.

Since $|\sin z| \geq |\sinh y|$, it happens that $|\sin z| \to \infty$ as $|y| \to \infty$.

8. Find all the complex numbers which satisfy the following:

(i) $\exp(z) = 1$ (ii) $\exp(z) = i$ (iii) $\exp(z - 1) = 1$.

Answer: This is the solution for part (iii); solutions for other parts are similar.

We need to find $z = x+iy$ such that $\exp(z-1) = \exp(x+iy-1) = \exp((x-1) + i\,y) = e^{(x-1)}\,(\cos y + i \sin y) = 1$.

That is,

$$e^{(x-1)} \cos y = 1 \qquad \text{and} \qquad e^{(x-1)} \sin y = 0 \ .$$

Observe that $e^{(x-1)} \neq 0$ for all $x \in \mathbb{R}$. Therefore, we need a y such that $\sin y = 0$ and $\cos y > 0$. This gives that $y = 2k\pi$ where $k \in \mathbb{Z}$.

When $y = 2k\pi$, $\cos y = 1$ and hence we need a x such that $e^{(x-1)} = 1$. It gives that $x = 1$. Hence, the points $z_k = (1, \ 2k\pi)$ where $k \in \mathbb{Z}$ are solutions of the equation $\exp(z - 1) = 1$.

9. Evaluate the following:

(i) $\log(3 - 2i)$ (ii) $\text{Log}\, i$ (iii) $(i)^{(-i)}$

Answer: (i) We know that if $z \neq 0$, then $\log(z) = \ln|z| \ + \ i \ \arg(z)$.

$$\log(3-2i) = \ln|3-2i| \ + \ i \ \arg(3-2i) = \ln|\sqrt{13}| \ + \ i \ (\alpha + 2n\pi)$$

where $\alpha = \tan^{-1}(-2/3)$ and $n \in \mathbb{Z}$. Therefore,

$$\log(3-2i) = \frac{1}{2} \ \ln(13) \ + \ i \ (\alpha + 2n\pi)$$

where $\alpha = \tan^{-1}(-2/3)$ and $n \in \mathbb{Z}$.

(ii) Recall if $z \neq 0$ then $\mathrm{Log}\,(z) = \ln|z| + i\mathrm{Arg}\,(z)$ where Arg denotes the principal value of the argument.

$$\mathrm{Log}\,(i) = \ \ln|i| \ + \ i \ \mathrm{Arg}\,(i) = \ \ln(1) \ + \ i \ \frac{\pi}{2} = \frac{i \ \pi}{2} \ .$$

(iii) We know for any $z \neq 0$ and w in $\mathbb{C}$, $z^w = \exp(w \log z)$. Therefore,

$$\begin{aligned}
(i)^{(-i)} &= \exp\left((-i)\log(i)\right) = \exp\left((-i)\left[\ln|i| \ + \ i \ \arg(i)\right]\right) \\
&= \exp\left((-i)\left[\ln(1) \ + \ i \ \left(\frac{\pi}{2} + 2n\pi\right)\right]\right) \\
&= \exp\left((-i)\left[i \ \left(\frac{\pi}{2} + 2n\pi\right)\right]\right) \\
&= \exp\left(\frac{\pi}{2} + 2n\pi\right)
\end{aligned}$$

where $n \in \mathbb{Z}$.

10. If γ is the boundary of the triangle with vertices at the points 0, $3i$ and -4 oriented in the counterclockwise direction then show that $\left| \int_C (e^z - \overline{z}) \ dz \right| \ \leq \ 60$.

Answer: Observe that the length of the curve γ is 12 i.e. $L = 12$.

Note that the function $|f(z)| = |e^z - \overline{z}| \leq |e^z| + |\overline{z}| = e^{\mathbb{R}e(z)} + |z|$. Therefore $|f(z)| \leq e^{\mathbb{R}e(z)} + |z| \leq 1 + |z| = 5$ for $z \in \gamma$ i.e. $M = 5$.

Now by ML inequality we have,

$$\left| \int_\gamma (e^z - \overline{z}) \ dz \right| \leq 5 \times 12 = 60.$$

11. Evaluate $\int_\gamma |z| \ \overline{z} \ dz$ where γ is the circle $|z| = 2$.

Answer: Use the formula

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Here $\gamma(t) = 2e^{it}$ for $t \in [0, \ 2\pi]$ and $f(z) = |z| \ \overline{z}$. This gives that $\gamma'(t) = 2i \ e^{it}$ for $t \in [0, \ 2\pi]$.

$$\int_\gamma |z| \ \overline{z} \ dz \ = \ \int_0^{2\pi} \left(2 \times 2e^{-it}\right) \ (2i \ e^{it}) \ dt = 8i \ \int_0^{2\pi} \ dt = 16\pi i.$$