MA201: Complex Analysis

Assignment 4: Model solutions/Hints

(Cauchy's theorem and Cauchy's integral formula)

July - November 2024

1. Show that $\int\limits_{\gamma} \frac{e^{az}}{z^2+1}dz = 2\pi i \sin a$, where $\gamma(t) = 2e^{it},\ t \in [0, 2\pi]$.

   Answer:

$$\int_{\gamma} \frac{e^{az}}{z^2+1}dz = \int_{\gamma} e^{az}\frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]dz = \frac{1}{2i}\left[\int_{\gamma}\frac{e^{az}}{z-i}dz - \int_{\gamma}\frac{e^{az}}{z+i}dz\right]$$

   By Cauchy integral formula,

$$\frac{1}{2i}\left[\int_{\gamma}\frac{e^{az}}{z-i}dz - \int_{\gamma}\frac{e^{az}}{z+i}dz\right] = 2\pi i \times \frac{1}{2i}[e^{ia} - e^{-ia}] = 2\pi i \sin a.$$

2. Evaluate $\int\limits_{0}^{2\pi} e^{e^{i\theta}} d\theta$.

   Answer: Put $e^{i\theta} = z$. Then $d\theta = \frac{dz}{iz}$. So

$$\int_0^{2\pi} e^{e^{i\theta}} d\theta = \int_{|z|=1} e^z \frac{dz}{iz} = 2\pi \text{ (by Cauchy integral formula).}$$

3. Let $f : \mathbb{C} \to \mathbb{C}$ be a function which is analytic on $\mathbb{C} \setminus \{0\}$ and bounded on $B(0, \frac{1}{2})$. Show that $\int_{|z|=R} f(z)dz = 0$ for all $R > 0$.

   Answer: By the deformation theorem, $\int_{|z|=R} f(z)dz = \int_{|z|=r} f(z)dz$ for every $r > 0$. Take $0 < r < \frac{1}{2}$. Given that f is bounded (by M say) on $B(0, \frac{1}{2})$, then by ML inequality

$$\left|\int_{|z|=r} f(z)dz\right| \leq M(2\pi r) \to 0 \text{ as } r \to 0.$$

4. Show that an entire function f satisfying $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all $z \in \mathbb{C}$ is a constant.

   Answer: It follows from the hypothesis that

$$f(z) = f(z+n) = f(z+im), \quad \text{for all} \quad z \in \mathbb{C}, \quad \text{and for all} \quad n, m \in \mathbb{Z}.$$

Let S be the rectangle with vertices $0, 1, 1+i$ and i. For any $z = x + iy \in \mathbb{C}$, there exits integers n and m and $z_0 = x_0 + iy_0 \in S$ such that,

$$z = x + iy = x_0 + n + i(y_0 + m) = z_0 + n + im.$$

This implies that $f(z) = f(z_0)$. In particular, $f(\mathbb{C}) = f(S)$. Since S is a compact set and f is a continuous function then $f(S)$ must be a bounded set. All together it implies that f is a bounded entire function. By Liouville's theorem, f is a constant function.

5. Let $g(z)$ be an analytic in $B(0,2)$. Compute $\int_{|z|=1} f(z)dz$ if

$$f(z) = \frac{a_k}{z^k} + \cdots + \frac{a_1}{z} + a_0 + g(z),$$

where a_i's are complex constants.

Answer: Since $a_0 + g(z)$ is analytic by Cauchy's theorem $\int_{|z|=1} [a_0 + g(z)]dz = 0$. Again $\frac{a_k}{z^k}$ has an antiderivative for $k \neq 1$ therefore, $\int_{|z|=1} \left[\frac{a_k}{z^k} + \cdots + \frac{a_2}{z^2}\right] dz = 0$. Therefore

$$\int_{|z|=1} f(z)dz = \int_{|z|=1} \frac{a_1}{z} dz = 2\pi i \times a_1.$$

6. Let f be an entire function such that $|f(0)| \leq |f(z)|$ for all $z \in \mathbb{C}$. Then either $f(0) = 0$ or f is constant.

Answer: If $f(0) = 0$ then the proof is trivial. If $f(0) \neq 0$ then $\left|\frac{1}{f(z)}\right| \leq \left|\frac{1}{f(0)}\right|$. So $\frac{1}{f(z)}$ is entire and bounded and by Liouville's theorem f is constant.

7. Whether primitive (anti-derivative) of $\frac{1}{z}$ exists on $\mathbb{C} \setminus \{0\}$? If NO, then specify the maximal domain in $\mathbb{C}$ where primitive exists.

Answer: If $\frac{1}{z}$ has primitive on $\mathbb{C} \setminus \{0\}$, then there exists some analytic function F_1 on $\mathbb{C} \setminus \{0\}$ such that $\frac{1}{z} = F_1'(z)$ on $\mathbb{C} \setminus \{0\}$, then by FTC, $\int_{|z|=r} \frac{1}{z} dz$ must be zero for any $r > 0$. However, we have $\int_{|z|=r} \frac{1}{z} dz = 2\pi i$.

It can be seen that the maximal domain of primitive of $\frac{1}{z}$ is D. If primitive exists on a larger domain D', then $\frac{1}{z} = F_1'(z)$ hold on D'. Let $a \in D' \cap \mathbb{R}_-$, then $\int_{|z|=|a|} \frac{1}{z} dz = 2\pi i$.

Alternative proof: Let $D = \mathbb{C} \setminus \mathbb{R}_-$, and $\mathbb{R}_- = \{z : \operatorname{Re} z \leq 0,\, Im(z) = 0\}$. Then we have $\dfrac{1}{z} = (Log\, z)'$ on D. That is, $(F_1 - Log)' = 0$ on D. This implies $F_1(z) = Log\, z + c$ on D. Since F_1 is primitive of $\frac{1}{z}$ on $\mathbb{C} \setminus \{0\}$, it implies that F_1 should be continuous on $\mathbb{C} \setminus \{0\}$, which is a contradiction.

8. Show that for $m \neq -1$, the z^m has primitive on $\mathbb{C} \setminus \{0\}$.

Answer: Do it yourself.